# A Quadratically Convergent $0(\sqrt{n}L)$-Iteration Algorithm for Linear Programming

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# A Quadratically Convergent $O(\sqrt{n}L)$-Iteration Algorithm for Linear Programming

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## Abstract

Recently, Ye et al. [15] proposed a large step modification of the Mizuno-Todd-Ye predictor-corrector interior-point algorithm for linear programming. They demonstrated that the large-step algorithm maintains the $O(\sqrt{n}L)$-iteration complexity while exhibiting superlinear convergence of the duality gap to zero under the assumption that the iteration sequence converges, and quadratic convergence of the duality gap to zero under the assumption of nondegeneracy. In this paper we establish the quadratic convergence result without any assumption concerning the convergence of the iteration sequence or nondegeneracy. This surprising result, to our knowledge, is the first instance of polynomiality and superlinear (or quadratic) convergence for an interior-point algorithm which does not assume the convergence of the iteration sequence or nondegeneracy.

**Key words:** Linear programming, primal and dual, superlinear and quadratic convergence, polynomiality

**Abbreviated title:** A quadratically convergent $O(\sqrt{n}L)$ algorithm for LP

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## 1. Introduction

Consider the primal linear program (LP):

$$\min\ c^T x$$
$$s.t.\quad Ax = b,\ x \geq 0,$$

and its dual (LD):

$$\max\ b^T y$$
$$s.t.\quad A^T y + s = c,\ s \geq 0,$$

where $A \in \mathbf{R}^{m\times n}$, $c \in \mathbf{R}^n$, and $b \in \mathbf{R}^m$. We say that $s$ is feasible for (LD) if there exists $y$ such that $(y, s)$ is feasible for (LD). Recall that a feasible point is said to be strictly feasible if it is feasible and positive. We say that $(x, s)$ is a (strictly) feasible pair for (LP) and (LD) if $x$ is (strictly) feasible for (LP) and $s$ is (strictly) feasible for (LD). It is well-known that for a feasible pair $(x, s)$ the duality gap is given by $x^T s$. Hence a feasible pair $(x^*, s^*)$ is optimal if and only if

$$x_j^* s_j^* = 0 \quad \text{for} \quad j = 1, 2, ..., n.$$

Moreover, consider a sequence of strictly feasible pairs $\{(x^k, s^k)\}$ such that the duality gap sequence $(x^k)^T s^k \to 0$. Then we say that this duality gap sequence converges Q-superlinearly to zero if

$$\lim_{k\to\infty} \frac{(x^{k+1})^T s^{k+1}}{(x^k)^T s^k} = 0,$$

and Q-quadratically to zero if

$$\limsup_{k\to\infty} \frac{(x^{k+1})^T s^{k+1}}{((x^k)^T s^k)^2} < +\infty.$$

In the context of the present work it is important to emphasize that the notions of convergence, superlinear convergence, or quadratic convergence of the duality gap sequence in no way require the convergence of the iteration sequence $\{(x^k, s^k)\}$. Of course, from Hoffman's lemma [4] it follows that in a particular sense the iteration sequence converges to the optimal solution set with the corresponding R-rate.

Recently, there has been an exciting outbreak of activity in the area of constructing primal-dual interior-point algorithms for either the linear programming problem (LP), or the linear complementarity problem (LCP) with a strict complementarity solution, that are demonstrably superlinearly or quadratically convergent. For LP, these works include Zhang et al. [16], [17], Ye et al. [15] and McShane [8]. For LCP, these works include Zhang et al. [18], Kojima et al. [6], and Ji et al. [5].

For the moment assume that the iteration sequence $\{(x^k, s^k)\}$ has been generated by an interior-point algorithm. Consider the following assumptions:

A0 a strictly feasible pair $(x^0, s^0)$ exists;

A1 the iteration sequence $\{(x^k, s^k)\}$ converges;

A2 the linear program is nondegenerate.

We intend A0 and A1 to apply to both LP and LCP. It is known that A2 implies A1 when the duality gap converges to zero. Note that A0 is assumed by all of the existing primal-dual interior-point algorithms. Concerning the results mentioned above, all of the superlinear convergence results assumed A1; while all of the quadratic convergence results assumed A2. Perhaps the most striking theoretical results obtained so far can be cataloged as follows:

- $O(nL)$ iteration complexity and superlinear convergence assuming A1 (Zhang and Tapia [16] for LP and Ji et al. [5] for LCP) or quadratic convergence assuming A2 (Zhang and Tapia [16] for LP).
- $O(\sqrt{n}L)$ iteration complexity and superlinear convergence assuming A1 (Ye et al. [15] and McShane [8] for LP) or quadratic convergence assuming A2 (Ye et al. [15] for LP).

In these bounds $L$ represents the data length of the problem being solved.

Certainly, the global property of polynomiality and the local property of superlinearity are desirable. However, the degree to which A1 is restrictive is an open question at this time. Moreover, A2 is not at all realistic, since most real-world LP problems are degenerate.

In what follows we consider the large-step modification of the Mizuno-Todd-Ye predictor-corrector algorithm suggested by Ye et al. [15]. We show that this $O(\sqrt{n}L)$ iteration complexity algorithm actually gives quadratic convergence of the duality gap to zero without assuming either A1 or A2. (Of course we must assume A0 as usual.)

In Section 2 we review the large-step algorithm and collect several previously established estimates. Section 3 contains several technical results. Our main convergence result is given in Section 4, and a summary and concluding remarks are contained in Section 5.

## 2. The Predictor-Corrector Algorithm

In this section, we briefly describe the predictor-corrector LP algorithm [10]. We employ the notation $X = \text{diag}(x)$, $S = \text{diag}(s)$, etc. and let $\Omega$ denote the collection of all strictly feasible pairs $(x, s)$. Consider the neighborhood

$$\mathcal{N}(\alpha) = \{(x,s) \in \Omega : \|Xs/\mu - e\| \leq \alpha\},$$

where $\|.\|$ represents the $l_2$ norm, $\mu = x^T s/n$, $e$ is the vector of all ones, and $\alpha$ is a constant between 0 and 1.

To begin with choose a constant $0 < \beta \leq 1/4$ (a typical choice would be $1/4$). All search directions $d_x$, $d_s$, and $d_y$ will be defined as solutions of the following system of linear equations (Kojima et al. [7])

$$\begin{aligned} Xd_s + Sd_x &= \gamma\mu e - Xs \\ Ad_x &= 0 \\ A^T d_y + d_s &= 0. \end{aligned} \tag{1}$$

A typical iteration of the algorithm proceeds as follows. Given $(x^k, s^k) \in \mathcal{N}(\beta)$, we solve the system (1) with $(x,s) = (x^k, s^k)$ and $\gamma = 0$. Denote the resulting directions by $d_x^k$ and $d_s^k$. For some step length $\theta \geq 0$ let

$$x(\theta) = x^k + \theta d_x^k, \quad s(\theta) = s^k + \theta d_s^k,$$

and $\mu(\theta) = x(\theta)^T s(\theta)/n$. This is the predictor step. The specific choice for $\theta$ will be stated after we consider the following lemma that is essentially due to Mizuno et al. [10].

**Lemma 2.1.** If there exists a positive $\theta^k \le 1$ such that

$$\|X(\theta)s(\theta)/\mu(\theta) - e\| \le \alpha < 1 \quad \text{for all} \quad 0 \le \theta \le \theta^k, \tag{2}$$

then $(x(\theta^k), s(\theta^k)) \in \mathcal{N}(\alpha)$.

The proof of Lemma 2.1 follows directly from a continuity argument. Lemma 2.1 basically says that the strict feasibility of $(x(\theta^k), s(\theta^k))$ is guaranteed as long as (2) is satisfied. Thus, we can choose the *largest* step length $\theta^k \le 1$ such that (2) is satisfied for $\alpha = 2\beta$, and let

$$\hat{x}^k = x(\theta^k) \quad \text{and} \quad \hat{s}^k = s(\theta^k).$$

Now we solve the system (1) with $(x,s) = (\hat{x}^k, \hat{s}^k) \in \mathcal{N}(2\beta)$, $\mu = (\hat{x}^k)^T \hat{s}^k/n$, and $\gamma = 1$. Let $x^{k+1} = \hat{x}^k + d_x$ and $s^{k+1} = \hat{s}^k + d_s$. It has been proved that $(x^{k+1}, s^{k+1}) \in \mathcal{N}(\beta)$ (Lemma 3 [10]). This is the corrector step.

We are now in a position to state the algorithm

**Algorithm** (Large-step predictor-corrector)

By the large-step predictor-corrector algorithm we mean the Mizuno-Todd-Ye algorithm defined above with the step length given by the *largest* $\theta^k$ satisfying the conditions of Lemma 2.1 with $0 < \beta \le 1/4$ and $\alpha = 2\beta$.

The choice of $\theta^k$ in the algorithm requires one to find the roots of a quartic polynomial. From the proof of our main result we will see that the choice for $\theta^k$ need not be this involved and it suffices to choose $\theta^k$ as the lower bound given in Lemma 2.2 below, as was the case in Ye et al. [15]. These comments will be stated formally as a corollary to our main theorem.

Observe that the algorithm generates a sequence of feasible pairs satisfying

$$\|X^k s^k/\mu^k - e\| \le \beta \tag{3.1}$$

and

$$(x^{k+1})^T s^{k+1} = (\hat{x}^k)^T \hat{s}^k = (1-\theta^k)(x^k)^T s^k. \tag{3.2}$$

For convenience, in what follows let

$$\delta^k = D_x^k d_s^k / \mu^k.$$

From Mizuno et al. [10] (Lemmas 1, 2, and 4) we have that

$$\|\delta^k\| \le \sqrt{2}n/4, \tag{4.1}$$

and for $0 < \beta \le 1/4$

$$\theta^k \ge \min\left\{\frac{1}{2}, (\frac{1}{8\|\delta^k\|})^{1/2}\right\}. \tag{4.2}$$

Thus, these inequalities together with (3.2) imply that the iteration complexity of the large-step algorithm is $O(\sqrt{n}L)$. Note that the algorithm requires that the linear system (1) be solved twice at each iteration.

From relation (3.2), we see that if $(1-\theta^k) \to 0$ then the duality gap $(x^k)^T s^k$ converges to zero Q-superlinearly. Moreover, if $(1-\theta^k) = O((x^k)^T s^k)$ then the duality gap converges to zero Q-quadratically. In our convergence-rate analysis, as opposed to our complexity analysis, the big $O$ notation represents a quantity that may or may not depend on $n$ or $L$, the problem data, however this dependence will not be explicitly stated. The above lower bound in (4.2) for $\theta^k$, due to Mizuno et al., is not sufficient to demonstrate superlinear convergence since it is at most $1/2$. Thus, Ye et al. [15] derived the following lower bound for $\theta^k$.

**Lemma 2.2.** If $\theta^k$ is the largest $\theta^k$ satisfying the conditions of Lemma 2.1 with $\alpha = 2\beta$, then

$$\theta^k \ge \frac{2}{\sqrt{1 + 4\|\delta^k\|/\beta} + 1}.$$

Using the bound given in Lemma 2.2, Ye et al. [15] have proved that the predictor-corrector algorithm maintains the $O(\sqrt{n}L)$ iteration complexity, and also gives superlinear convergence under assumption A1 or quadratic convergence under assumption A2. In the next section we will show how to remove these assumptions and actually obtain quadratic convergence for general LP problems.

## 3. Technical Results

At the $k$th predictor step if $\theta^k$ is the largest $\theta^k$ satisfying the conditions of Lemma 2.1 with $\alpha = 2\beta$, then

$$\begin{aligned}
1-\theta^k &\le 1 - \frac{2}{\sqrt{1+4\|\delta^k\|/\beta}+1} \\
&= \frac{\sqrt{1+4\|\delta^k\|/\beta}-1}{\sqrt{1+4\|\delta^k\|/\beta}+1} \\
&= \frac{4\|\delta^k\|/\beta}{(\sqrt{1+4\|\delta^k\|/\beta}+1)^2} \\
&\le \|\delta^k\|/\beta.
\end{aligned} \tag{5}$$

Our goal is to prove that $\|\delta^k\| = O((x^k)^T s^k)$ without using assumption A1 or A2.

We first introduce several technical lemmas. For simplicity, we drop the index $k$ and recall the linear system during the predictor step

$$\begin{aligned}
Xd_s + Sd_x &= -Xs \\
Ad_x &= 0 \\
A^T d_y + d_s &= 0.
\end{aligned} \tag{6}$$

Let $\mu = x^T s/n$ and $z = Xs$. Then from (3.1) we must have

$$(1-\alpha)\mu \le z_j \le (1+\alpha)\mu \quad \text{for} \quad j = 1, 2, ..., n. \tag{7}$$

Define $D = X^{1/2}S^{-1/2}$ and denote by $\Pi_L$ the orthogonal projection onto the linear subspace $L$ of $\mathbf{R}^n$. Denote by $N(AD)$ and $R(DA^T)$ the null space of $AD$ and the range of $DA^T$, respectively. We shall estimate $\|d_x\|$ and $\|d_s\|$. Our present objective is to demonstrate that $\|d_x\| = O(\mu)$ and $\|d_s\| = O(\mu)$.

We start by characterizing the solution to (6).

**Lemma 3.1.** If $d_x$ and $d_s$ are obtained from the linear system (6), then

$$\begin{aligned}
d_x &= -D\Pi_{N(AD)} r, \\
d_s &= -D^{-1}\Pi_{R(DA^T)} r,
\end{aligned}$$

where $r = Z^{1/2}e$.

**Proof.** The proof is straightforward, e.g., see Adler and Monteiro [1]. ∎

It is well known that for every linear program, a unique partition $A = (B,\ N)$ exists such that the primal optimal facet is given by

$$\Omega_p = \{x_B : \ Bx_B = b,\ x_B \geq 0\}$$

and the dual optimal facet is given by

$$\Omega_d = \{(y, s_N) : \ c_B - B^T y = 0,\ s_N = c_N - N^T y \geq 0\}.$$

Strictly feasible solutions $x_B > 0$ and $s_N > 0$ exist on these optimal facets, respectively, and both facets are bounded under assumption A0. Here, we also use $B$ and $N$ to denote the partitioned column index sets. For all $k$, relation (3.1) implies that

$$\xi \leq x_j^k \leq 1/\xi \quad \text{for} \quad j \in B \tag{8.1}$$

and

$$\xi \leq s_j^k \leq 1/\xi \quad \text{for} \quad j \in N, \tag{8.2}$$

where $\xi < 1$ is a fixed positive number that is independent of $k$ (Güler and Ye [3]).

**Lemma 3.2.** If $d_x$ and $d_s$ are obtained from the linear system (6) and $\mu = x^T s/n$, then

$$\|(d_x)_N\| = O(\mu) \quad \text{and} \quad \|(d_s)_B\| = O(\mu).$$

**Proof.** From Lemma 3.1, we obtain

$$\begin{aligned} \|D^{-1}d_x\| &\leq \|\Pi_{N(AD)}\| \|r\| \\ &\leq \|r\| = O(\sqrt{\mu}). \end{aligned}$$

We have from relations (7) and (8)

$$\begin{aligned} \|(d_x)_N\| &= \|D_N D_N^{-1}(d_x)_N\| \\ &\leq \|D_N\| \|D_N^{-1}(d_x)_N\| \\ &\leq \|D_N\| O(\sqrt{\mu}) \\ &= \|Z_N^{1/2} S_N^{-1}\| O(\sqrt{\mu}) \\ &= O(\sqrt{\mu}) O(\sqrt{\mu}) = O(\mu). \end{aligned}$$

This proves that $\|(d_x)_N\| = O(\mu)$. The proof that $\|(d_s)_B\| = O(\mu)$ is similar. ∎

The proofs of $\|(d_x)_B\| = O(\mu)$ and $\|(d_s)_N\| = O(\mu)$ are more involved. Towards this end, we first note

$$\begin{aligned} x + d_x &= D\Pi_{R(DA^T)}r, \\ s + d_s &= D^{-1}\Pi_{N(AD)}r. \end{aligned} \tag{9}$$

This is because from the first equation of (6) we have

$$\begin{aligned} S(x + d_x) &= -Xd_s \\ X(s + d_s) &= -Sd_x. \end{aligned}$$

Thus,

$$\begin{aligned} x + d_x &= -(XS^{-1})d_s = -D^2 d_s \\ s + d_s &= -(SX^{-1})d_x = -D^{-2}d_x, \end{aligned}$$

which gives relation (9).

The following lemma is essentially due to Adler and Monteiro [1] (also see Sonnevend et al. [11] and Witzgall et al. [13]).

**Lemma 3.3.** If $d_x$ and $d_s$ are obtained from the linear system (6), then $(d_x)_B$ is the solution to the (weighted) least-squares problem

$$\begin{aligned} \min_u \quad & (1/2)\|D_B^{-1}u\|^2 \\ s.t. \quad & Bu = -N(d_x)_N, \end{aligned} \tag{10.1}$$

and $(d_s)_N = -N^T d_y$ where $d_y$ is a solution to the (weighted) least-squares problem

$$\begin{aligned} \min_v \quad & (1/2)\|D_N N^T v\|^2 \\ s.t. \quad & B^T v = -(d_s)_B. \end{aligned} \tag{10.2}$$

**Proof.** From (9), we see that

$$x_B + (d_x)_B \in R(D_B^2 B^T). \tag{11}$$

Since $s_B^* = 0$ for all optimal $s^*$, we must have $c_B \in R(B^T)$. Thus,

$$s_B = c_B - B^T y \in R(B^T),$$

which implies that

$$x_B = D_B^2 s_B \in R(D_B^2 B^T). \tag{12}$$

From (11) and (12) we have

$$(d_x)_B \in R(D_B^2 B^T).$$

Moreover, $(d_x)_B$ satisfies the equation

$$B(d_x)_B = -N(d_x)_N.$$

Thus, $(d_x)_B$ satisfies the Karush-Kuhn-Tucker conditions for the least squares problem (10.1).

Since $AD^2(s + d_s) = -Ad_x = 0$ and $AD^2 s = Ax = b$, it follows that

$$-b = AD^2 d_s = BD_B^2(d_s)_B + ND_N^2(d_s)_N. \tag{13}$$

Also, since $x_N^* = 0$ for all optimal $x^*$, we have $Bx_B^* = b$ implying $b \in R(B)$. Therefore, relation (13) implies

$$ND_N^2 N^T d_y = -ND_N^2(d_s)_N \in R(B).$$

Moreover, $d_y$ satisfies the equation

$$B^T d_y = -(d_s)_B.$$

Thus, $d_y$ satisfies the Karush-Kuhn-Tucker conditions for the least squares problem (10.2). ∎

**Theorem 3.1.** If $d_x$ and $d_s$ are obtained from the linear system (6) and $\mu = x^T s/n$, then

$$\|(d_x)_B\| = O(\mu) \quad \text{and} \quad \|(d_s)_N\| = O(\mu).$$

**Proof.** Since the least-squares problem (10.1) is always feasible, there must be a *feasible* $\bar{u}$ such that

$$\|\bar{u}\| = O(\|(d_x)_N\|),$$

which together with Lemma 3.2 implies

$$\|\bar{u}\| = O(\mu).$$

Furthermore, from Lemma 3.3 and relations (7) and (8)

$$\begin{aligned}
\|(d_x)_B\| &= \|D_B D_B^{-1}(d_x)_B\| \\
&\leq \|D_B\|\|D_B^{-1}(d_x)_B\| \\
&\leq \|D_B\|\|D_B^{-1}\bar{u}\| \\
&\leq \|D_B\|\|D_B^{-1}\|\|\bar{u}\| \\
&= \|Z_B^{-1/2}X_B\|\|Z_B^{1/2}X_B^{-1}\|\|\bar{u}\| \\
&\leq \|Z_B^{-1/2}\|\|X_B\|\|Z_B^{1/2}\|\|X_B^{-1}\|\|\bar{u}\| \\
&= O(\|\bar{u}\|) = O(\mu).
\end{aligned}$$

Similarly, we can prove the second statement of the theorem. ∎

## 4. Quadratic Convergence without Assumption A1 or A2

Theorem 3.1 indicates that at the $k$th predictor step, $d_x^k$ and $d_s^k$ satisfy

$$\|(d_x^k)_B\| = O(\mu^k) \quad \text{and} \quad \|(d_s^k)_N\| = O(\mu^k) \tag{14}$$

where $\mu^k = (x^k)^T s^k/n$. We are now in a position to state our main result.

**Theorem 4.1.** Let $\{(x^k, s^k)\}$ be the sequence generated by the Algorithm. Then, with constants $0 < \beta \leq 1/4$ and $\alpha = 2\beta$

(i) the Algorithm has iteration complexity $O(\sqrt{n}L)$;

(ii) $1 - \theta^k = O((x^k)^T s^k)$;

(iii) $(x^k)^T s^k \to 0$ Q-quadratically.

**Proof.** The proof of (i), i.e., the $O(\sqrt{n}L)$-iteration complexity of the algorithm follows from inequalities (3.2), (4.1) and Lemma 2.2, which give

$$(x^{k+1})^T s^{k+1} \leq (1 - \frac{2}{\sqrt{1+\sqrt{2}n/\beta} + 1})(x^k)^T s^k.$$

This also establishes

$$\lim_{k\to\infty} \mu^k = 0.$$

For every $j$, at the $k$th predictor step (i.e. $\gamma = 0$), it follows from system (1) that

$$\frac{(d_x^k)_j}{x_j^k} + \frac{(d_s^k)_j}{s_j^k} = -1, \quad j = 1, 2, ..., n. \tag{15}$$

From (8) and (14) we have

$$\left|\frac{(d_x^k)_j}{x_j^k}\right| = O(\mu^k) \quad \text{for} \quad j \in B \tag{16.1}$$

and

$$\left|\frac{(d_s^k)_j}{s_j^k}\right| = O(\mu^k) \quad \text{for} \quad j \in N. \tag{16.2}$$

Relation (16) implies that

$$\lim_{k\to\infty} \frac{(d_x^k)_j}{x_j^k} = 0 \quad \text{for} \quad j \in B$$

and

$$\lim_{k\to\infty} \frac{(d_s^k)_j}{s_j^k} = 0 \quad \text{for} \quad j \in N,$$

which together with (15) implies that

$$\lim_{k\to\infty} \frac{(d_s^k)_j}{s_j^k} = -1 \quad \text{for} \quad j \in B \tag{17.1}$$

and

$$\lim_{k\to\infty} \frac{(d_x^k)_j}{x_j^k} = -1 \quad \text{for} \quad j \in N \tag{17.2}$$

Thus, relations (16) and (17) imply that for sufficiently large $k$

$$\left|\frac{(d_x^k)_j(d_s^k)_j}{x_j^k s_j^k}\right| = O(\mu^k) \quad \text{for} \quad j = 1, 2, ..., n. \tag{18}$$

Furthermore, note from (3.1) that

$$(1-\beta)\mu^k \le x_j^k s_j^k \le (1+\beta)\mu^k \quad \text{for} \quad j = 1, 2, ..., n. \tag{19}$$

Therefore, inequalities (18) and (19) imply that

$$\|\delta^k\| = \|D_x^k d_s^k/\mu^k\| = O(\|(X^k S^k)^{-1} D_x^k d_s^k\|) \leq O(\sqrt{n}\mu^k) = O((x^k)^T s^k),$$

which together with (5) establishes (ii). From (3.2) we see that (ii) implies (ii). This proves the theorem. ∎

Interestingly, the behavior of $(X^k)^{-1}d_x^k$ and $(S^k)^{-1}d_s^k$ in our Theorem was discussed by Tapia [12] in (1980) for complementarity problems where the strict complementarity solution is unique. He used these two vectors as the basis of an indicator theory for identifying variables which are zero at the solution. See also El-Bakry et al. [2]. These two vectors were also used by Mehrotra and Ye [9] as a criterion for identifying the partition $B$ and $N$.

The following corollary formally states that we do not need to choose the largest step in the predictor step, but only a sufficiently large step. Thus, we are no longer required to find the zeros of a quartic polynomial.

**Corollary 4.1.** If the predictor-corrector algorithm is adapted with $\theta^k$ given by the lower bound in Lemma 2.2, then Theorem 4.1 also holds for the modified algorithm.

## 5. Summary and Concluding Remarks

Recently, Mizuno et al. [10] proposed a predictor-corrector interior-point algorithm for linear programming. They demonstrated that the algorithm possessed $O(\sqrt{n}L)$ iteration complexity. More recently, Ye et al. [15] proposed a large-step modification of the algorithm and proved that the large-step algorithm, while maintaining $O(\sqrt{n}L)$ iteration complexity, exhibited superlinear convergence of the duality gap sequence to zero under the assumption that the iteration sequence converged, and exhibited quadratic convergence of the duality gap sequence to zero under the assumption of nondegeneracy. In this paper we have established the surprising result that the large-step predictor-corrector algorithm actually exhibits quadratic convergence of the duality gap to zero without the assumption of nondegeneracy or even the assumption that the iteration sequence converges. This result is the first instance of a demonstration of polynomiality and superlinear (or quadratic) convergence for an interior-point method which does not assume the convergence of the iteration sequence or nondegeneracy. We note that each iteration in the predictor-corrector algorithm requires the solutions of two linear systems—one in the predictor step and one in the corrector step.

Although the iteration sequence $\{(x^k, s^k)\}$ may not be convergent, it is a consequence of Hoffman's lemma [4] that the sequence $\{x^k\}$ converges $R$-quadratically to the primal optimal set $\Omega_p$. The same is true for the sequence $\{s^k\}$ if we write the dual linear program and its optimal solution set in terms of $s$ alone.

While seemingly quadratic convergence has often been observed in practice for primal-dual interior-point methods applied to degenerate problems, its effectiveness is compromised by the use of finite-precision arithmetic to solve the necessarily ill-conditioned linear systems. Hence our current result may have only theoretical value. In this context the finite termination procedures of Ye [14] and Mehrotra and Ye [9] are of value in obtaining an optimal solution.

It has been observed in practice that the $O(\sqrt{n}L)$ algorithms are generally less effective than are some of the $O(nL)$ algorithms (or other non-polynomial algorithms). Zhang, Tapia and Dennis [17] argued that several of these $O(\sqrt{n}L)$ algorithms possess

particularly poor Q-convergence properties, i.e., they exhibit Q-linear convergence with convergence constants near 1 for large $n$. Therefore, some researchers may have embraced the belief that the $O(\sqrt{n}L)$ were less effective because none of them could achieve superlinear convergence. Now, we have constructed an $O(\sqrt{n}L)$ algorithm that has superior (actually optimal) asymptotic convergence when compared to the $O(nL)$ algorithms for degenerate problems. If, perchance, numerical experimentation still favors the $O(nL)$ algorithms, then we must conclude that their advantage is not due to their asymptotic behavior, but to some other, as yet unexplained, phenomenon.

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