Parallel Domain Decomposition Method for Mixed Finite Elements for Elliptic Partial Differential Equations

Lawrence C. Cowsar
Mary F. Wheeler

Center for Research on Parallel Computation
Rice University
P.O. Box 1892
Houston, TX 77251-1892

PARALLEL DOMAIN DECOMPOSITION METHOD FOR MIXED FINITE ELEMENTS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

LAWRENCE C. COWSAR* AND MARY F. WHEELER*

Abstract. In this paper we develop a parallel domain decomposition method for mixed finite element methods. This algorithm is based on a procedure first formulated by Glowinski and Wheeler for a two subdomain problem. This present work involves extensions of the above method to an arbitrary number of subdomains with an inner product modification and multilevel acceleration. Both Neumann and Dirichlet boundary conditions are treated. Numerical experiments performed on the Intel iPSC/860 Hypercube are presented and indicate that this approach is scalable and fairly insensitive to variation in coefficients.

---

* Mathematical Sciences Dept., Rice University, Houston, TX 77251.

1. Introduction. Mixed finite methods are known to be important in modeling flow in porous media. In a mixed finite element formulation both pressure and velocities are approximated simultaneously. In addition, approximating spaces may be chosen so that mass is conserved cell by cell. Application of these methods to porous media problems was first proposed and analyzed in [6, 7] and computational results presented in [3].

We note that standard petroleum industrial simulators employ cell centered finite difference methods for approximating the pressure equation. Russell and Wheeler in [14] first showed that these difference methods are in fact the lowest order Raviart–Thomas mixed finite element method with special numerical quadrature. A second remark is that the approximation of the pressure equation is the most costly portion of the numerics in a reservoir simulator. Evidence of this is easily discernible from the petroleum engineering literature which contains a large number of papers on linear solvers.

Because flow in porous media problems are computationally intensive, many government and industrial laboratories are investigating parallel algorithms for reservoir and contaminant transport simulation on distributive memory machines like the Intel RX, the NCUBE, and the CM-2. The benefits include both the opportunity to solve problems cheaper and the ability to solve problems otherwise too large. Present day simulators frequently employ grid spacings at least 200 times too large for desired accuracy. Moreover, in porous media simulations many realizations are needed because of the stochastic nature of the coefficients or heterogeneities.

Computer engineers indicate that by the end of the decade supercomputers will be teraflop machines with thousands of processors. With this in mind we characterize a scalable algorithm for distributed memory machines as an algorithm which requires minimal internode communication and in which the number of iterations remains nearly constant as the mesh decreases and the number of domains increases (scaling).

In the First International Domain Decomposition meeting held in Paris, France, Glowinski and Wheeler [10] introduced two domain decomposition algorithms for mixed finite elements methods. A key result in [10] was the formulation of the matching conditions at the interfaces of the subdomains as variational problems defined over trace spaces. In Method 1, fluxes were assumed to match and iterations were performed to match pressure. This in fact appears to be more natural for mixed finite element formulations since flux boundary conditions are essential. The interface problem arising from Method 1 was solved by a conjugate gradient algorithm resulting in a $0(1/\sqrt{h})$ number of iterations to achieve convergence.

Method 2 is the dual of Method 1; pressures are assumed to match and iterations are performed to match fluxes. In [10] only a two domain problem was considered.

In the Third International Domain Decomposition, Glowinski, Kinton, and Wheeler [9] defined a multilevel acceleration of Method 1. Numerical experiments carried out on a sequential machine indicated that the number of V cycles was practically independent of h despite the fact that the dimension of the interface problem is growing that $1/h$.

A difficulty in the parallelization of Method 1 on a distributed memory machine is the adjustment of the pressure solution over the subdomains. Each evaluation of the bilinear form requires the solution of a tridiagonal system of order at most $M-1$, M being the number of subdomains. The order of the linear system varies depending on the boundary conditions imposed and the decomposition employed. In Method 2 the pressure adjustment by a constant only arises for Neumann problems, and there is no linear system to solve.

In Section 2 we define Method 2 for arbitrary number of domains and prove the strong ellipticity of the resulting bilinear form. We also prove that the condition number of the bilinear form is $0(1/h)$. Results of numerical experiments carried out on an Intel RX will be discussed in Section 3 using a generalization of the Raviart Thomas spaces for domains which are unions of rectangles. Conclusions are presented in Section 4.

2. Domain Decomposition Method for Mixed Finite Element Methods.
We consider the elliptic partial differential equation

$$(2.1) \qquad \mathbf{u} = -A\nabla p \qquad \text{in } \Omega,$$

$$(2.2) \qquad \nabla \cdot \mathbf{u} = f \qquad \text{in } \Omega,$$

$$(2.3) \qquad p = q \qquad \text{on } \hat{\Gamma},$$

$$(2.4) \qquad \mathbf{u} \cdot \mathbf{n} = g \qquad \text{on } \Gamma^*,$$

where Ω is a bounded connected subset of $\Re^2$, $\Gamma^* \cap \hat{\Gamma} = \emptyset$, $\Gamma^* \cup \hat{\Gamma} = \partial\Omega$, and A is symmetric and uniformly elliptic.

For convenience we restrict our attention to two dimensional problems; extensions to three dimensional problems are straightforward.

Let V and W be finite dimensional subspaces such that

$$V \quad \subset \quad H(\Omega,\ div) \ = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \nabla \cdot \mathbf{v} \in L^2(\Omega)\},$$

$$V^* \quad = \quad V \cap \{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{n} = 0 \ \text{on } \Gamma^*\},$$

$$W \quad \subset \quad L^2(\Omega), \ \text{such that } div(V) \subset W.$$

Let $(\phi, \psi) = \int_\Omega \phi\psi\, dx$. The mixed finite element approximation to $\{\mathbf{u}, p\}$ is given by $\{\mathbf{U}, P\} \in V \times W$ satisfying

$$(2.5) \qquad a(\mathbf{U}, P, \mathbf{v}) \equiv (A^{-1}\mathbf{U}, \mathbf{v}) - (P, \nabla \cdot \mathbf{v}) = \int_{\hat{\Gamma}} q\mathbf{v} \cdot \mathbf{n}\, ds, \quad \mathbf{v} \in V^*,$$

3

$$(2.6) \qquad (\nabla \cdot \mathbf{u}, w) = (f, w), \quad w \in W,$$

$$(2.7) \qquad \int_{\Gamma_*} (\mathbf{U} \cdot \mathbf{n} - g) \mathbf{v} \cdot \mathbf{n} \, ds = 0, \quad \mathbf{v} \in V.$$

Equations (2.5)–(2.7) are a weak formulation of (2.1)–(2.4) obtained by multiplying (2.1) by $A^{-1}\mathbf{v}$ and integrating by parts and by multiplying (2.2) by w and integrating.

Before describing the domain decomposition mixed finite element Method 2 for M subdomains we establish some notation.

Let $\Omega = \cup_{j=1}^{M} \Omega_j$ with $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Denote by $\partial \Omega_i$ the boundary of Ω_i, $\gamma_{ij} = \gamma_{ji} = \partial \Omega_i \cap \partial \Omega_j$ and $\mathbf{n}_{ij}$ the normal to γ_{ij} pointing outward from Ω_i. We set $\gamma = \cup_{i,j} \gamma_{ij}$; we only consider those γ_{ij} whose measure is positive. Let V_i and W_i be restrictions of V and W respectively to Ω_i.

Set

$$
\begin{aligned}
(\phi, \psi)_i &= \int_{\Omega_i} \phi \psi \, dx \quad \phi, \psi \in L^2(\Omega_i), \\
V_i^0 &= V_i \cap \{ \mathbf{v} \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega_i \}, \\
V_i^* &= V_i \cap \{ \mathbf{v} \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega_i \cap \Gamma^* \}, \\
V_\gamma &= \{ \mu \mid \mu \epsilon L^2(\gamma) \text{ and } \mu = \mathbf{v} \cdot \mathbf{n}_{ij}, \mathbf{v} \in V, \\
& \qquad \mathbf{meas}(\partial \Omega_i \cap \partial \Omega_j) > 0, i, j = 1, 2, \ldots, M \}, \\
V_\gamma^* &= \begin{cases} V_\gamma & \text{if } \mathbf{meas}(\hat{\Gamma}) \neq 0 \\ V_\gamma \cap \{ z \mid \int_\gamma z \, ds = 0 \} = V_\gamma/\mathbb{R} & \text{otherwise.} \end{cases}
\end{aligned}
$$

Also we denote by $[\mathbf{U} \cdot \mathbf{n}]$ the jump of the flux of $\mathbf{U} \in \cup_{i=1}^{M} V_i$ on γ.

Assume that we have the following regularity assumption (R): Let $\Omega^* = \bigcup_{j \in J} \Omega_j$ for some $J \subset \{1, \ldots, M\}$ such that Ω^* is connected. We assume that if $\mathbf{meas}(\partial \Omega^* \cap \Gamma^*) > 0$ ($\mathbf{meas}(\partial \Omega^* \cap \Gamma^*) = 0$), then there exists a unique $\phi \in H^{1+\epsilon}(\Omega^*)$ ($\phi \in H^{1+\epsilon}(\Omega^*)/\Re$) for some $\epsilon > 0$ satisfying

$$
(2.8) \qquad
\begin{aligned}
-\nabla \cdot A \nabla \phi &= 0 \quad \text{in } \Omega^*, \\
\phi &= 0 \quad \text{on } \partial \Omega^* \cap \hat{\Gamma}, \\
A \nabla \phi \cdot \mathbf{n} &= \begin{cases} 0 & \text{on } \partial \Omega^* \cap \Gamma^*, \\ z & \text{on } \partial \Omega^* \cap \gamma. \end{cases}
\end{aligned}
$$

where $z \in L^2(\partial \Omega^* \cap \gamma)$ and $\int_{\partial \Omega_i \cap \gamma} z \, ds = 0$ if $\mathbf{meas}(\partial \Omega^* \cap \Gamma^*) = 0$.

Define $a_i : V_i \times W_i \times V_i \to \Re$ by

$$a_i(\mathbf{U}, P, \mathbf{v}) = (A^{-1}\mathbf{U}, \mathbf{v})_i - (P, \nabla \cdot \mathbf{v})_i.$$

Proceeding as in Glowinski and Wheeler [10], we have that (2.5)–(2.7) is equivalent to the following problem: Find $\{\mathbf{U}(\lambda), P(\lambda), \lambda\}$ where $\mathbf{U}(\lambda)|_{\Omega_i} \in V_i$, $P(\lambda)|_{\Omega_i} \in W_i$,

$\lambda \in V_\gamma^*$ satisfying

$$(2.9) \qquad a_i(\mathbf{U}(\lambda), P(\lambda), \mathbf{v}) = \int_{\partial\Omega_i \setminus \partial\Omega_i \cap \Gamma^*} \lambda \mathbf{v} \cdot \mathbf{n}\, ds$$

$$+ \int_{\partial\Omega_i \cap \hat{\Gamma}} q\mathbf{v} \cdot \mathbf{n}\, ds\,, \quad \mathbf{v} \in V_i^*\,,$$

$$(2.10) \qquad (\nabla \cdot \mathbf{U}(\lambda), w)_i = (f, w)_i\,, \quad w \in W_i\,,$$

$$(2.11) \qquad \int_{\Gamma^*} (\mathbf{U}(\lambda) \cdot \mathbf{n} - g)\mathbf{v} \cdot \mathbf{n}\, ds = 0\,, \quad \mathbf{v} \in V\,,$$

and

$$(2.12) \qquad \int_\gamma [\mathbf{U}(\lambda) \cdot \mathbf{n}]\mu\, ds = 0\,, \quad \mu \in V_\gamma^*\,.$$

One can show that if $\{\mathbf{U}(\lambda), P(\lambda), \lambda\}$ satisfy (2.9)–(2.12), then $\{\mathbf{U}(\lambda), P(\lambda)\}$ satisfy (2.5)–(2.7).

Let $\bar{\lambda} \in V_\gamma^*$ and define $\mathbf{U}(\bar{\lambda})$ and $P(\bar{\lambda})$ by (2.9)–(2.11). Setting $\lambda^* = \lambda - \bar{\lambda}$, $\mathbf{U}(\lambda^*) = \mathbf{U} - \mathbf{U}(\bar{\lambda})$, and $P(\lambda^*) = P - P(\bar{\lambda})$, we see that for $i = 1, \ldots, M$,

$$(2.13) \qquad a_i(\mathbf{U}(\lambda^*), P(\lambda^*), \mathbf{v}) = \int_{\partial\Omega_i \setminus \partial\Omega_i \cap \Gamma^*} \lambda^* \mathbf{v} \cdot \mathbf{n}\, ds\,, \quad \mathbf{v} \in V_i^*\,,$$

$$(2.14) \qquad \int_{\Gamma^*} \mathbf{U}(\lambda^*) \cdot \mathbf{n}\ \mathbf{v} \cdot \mathbf{n}\, ds = 0\,, \quad \mathbf{v} \in V\,,$$

$$(2.15) \qquad (\nabla \cdot \mathbf{U}(\lambda^*), w)_i = 0\,, \quad w \in W_i\,,$$

and

$$(2.16) \qquad \int_\gamma [\mathbf{U}(\lambda^*) \cdot \mathbf{n}]\mu\, ds = -\int_\gamma [\mathbf{U}(\bar{\lambda}) \cdot \mathbf{n}]\mu\, ds\,, \quad \mu \in V_\gamma^*\,.$$

Define the bilinear form $b : V_\gamma^* \times V_\gamma^* \to \Re$ by

$$(2.17) \qquad b(\lambda^*, \mu) = \int_\gamma [\mathbf{U}(\lambda^*) \cdot \mathbf{n}]\mu\, ds\,,$$

where $\mathbf{U}(\lambda^*)$ is defined by (2.13)–(2.15); further define the linear functional $L : V_\gamma* \to \Re$ by

$$(2.18) \qquad L(\mu) = -\int_\gamma [\mathbf{U}(\bar{\lambda}) \cdot \mathbf{n}]\mu\, ds\,.$$

To determine $\mathbf{U} = \mathbf{U}(\lambda^*) + \mathbf{U}(\bar{\lambda})$ it suffices to solve the following interface problem:

Find $\lambda^* \in V_\gamma^*$ such that

$$(2.19) \qquad b(\lambda^*, \mu) = L(\mu)\,, \quad \mu \in V_\gamma^*\,,$$

where $\{\mathbf{U}(\lambda^*), P(\lambda^*)\}$ satisfy (2.13)–(2.15). This procedure is known as Method 2.

We now prove the following results on the bilinear form $b(\cdot, \cdot)$.

THEOREM 2.1. *The bilinear form $b : V_\gamma^* \times V_\gamma^* \to \mathbb{R}$ is symmetric and positive definite if* $\text{meas}(\hat{\Gamma}) \neq 0$, *and is positive definite over* $V_\gamma^*/\mathbb{R}$ *if* $\text{meas}(\hat{\Gamma}) = 0$. *The bilinear form is strongly elliptic over the respective spaces.*

Proof. We first establish the symmetry of b. We have

$$b(\lambda, \mu) = \int_\Omega [\mathbf{U}(\lambda) \cdot \mathbf{n}]\mu \, ds, \quad \mu \in V_\gamma^*,$$

where $\{\mathbf{U}(\lambda), P(\lambda)\}$ are defined by (2.13)–(2.15) with $\lambda^* = \lambda$. We define the pair $\{\mathbf{U}(\mu), P(\mu)\}$ also by (2.13)–(2.15) for $\mu \in V_\gamma^*$. Thus,

$$a_i(\mathbf{U}(\mu), P(\mu), \mathbf{U}(\lambda)) = \int_{\partial\Omega_i \backslash \partial\Omega_i \cap \Gamma^*} \mu \mathbf{U}(\lambda)_{ds} \cdot \mathbf{n} \, .$$

Summing (2.20) for $i = 1, \ldots, M$ we obtain

$$
\begin{aligned}
(2.20) \qquad b(\lambda, \mu) &= \sum_{i=1}^M \int_{\Omega_i} A^{-1}\mathbf{U}(\lambda) \cdot \mathbf{U}(\mu) \, dx \\
&= b(\mu, \lambda) \, ,
\end{aligned}
$$

which immediately implies b is symmetric over V_γ^* and positive semi-definite.

We now show that b is positive definite if $\text{meas}(\hat{\Gamma}) > 0$, i.e. if $b(\lambda, \lambda) \equiv 0$, then $\lambda \equiv 0$.

Let Ω_ℓ be a domain in which $\text{meas}(\partial\Omega_\ell \cap \hat{\Gamma}) > 0$. Define $\{\psi_\ell, \phi_\ell\}$ to the solution of the elliptic partial differential equation given by

$$
(2.21) \qquad
\begin{aligned}
\psi_\ell &= -A\nabla\phi_\ell \quad \text{in } \Omega_\ell, \\
\nabla \cdot \psi_\ell &= 0 \quad \text{in } \Omega_\ell, \\
\phi_\ell &= 0 \quad \text{on } \partial\Omega_\ell \cap \hat{\Gamma}, \\
\psi_\ell \cdot \mathbf{n} &= \begin{cases} 0 & \text{on } \partial\Omega_\ell \cap \Gamma^*, \\ \lambda & \text{on } \partial\Omega_\ell \cap \gamma. \end{cases}
\end{aligned}
$$

By the regularity assumption (**R**) we have that $\phi_\ell \in H^{1+\epsilon}(\Omega_\ell)$. Define $\{\psi_\ell^*, \phi_\ell^*\} \in V_\ell \times W_\ell$ to be the mixed finite element approximation of (2.21) defined by (2.5)–(2.7) with appropriate changes in domain definition. Applying (2.13) and noting that $\mathbf{U}(\lambda) \equiv 0$, we obtain

$$\int_{\partial\Omega_\ell \cap \gamma} (\lambda)^2 \, ds = -\int_{\Omega_\ell} P(\lambda)\nabla \cdot \psi_\ell^* \, dx = 0 \, .$$

Thus $\lambda = 0$ on $\partial\Omega_\ell \cap \gamma$.

We now consider an adjacent domain to Ω_ℓ in which the boundary intersection has positive measure. Call such a domain Ω_k. Define ϕ_k to be the solution of the boundary value problem (2.21) with the modification that $\phi_k = 0$ on $\partial\Omega_k \cap \partial\Omega_\ell \cap \gamma$. Repeating the above argument we note that $\lambda = 0$ on $\partial\Omega_k \cap \gamma$.

6

We now exhaust the number of domains and conclude that $\lambda = 0$ on γ.

For the pure Neumann problem in which $\mathbf{meas}(\hat{\Gamma}) = 0$ we consider the following problems:

$$(2.22) \qquad \begin{aligned} \psi_\ell &= -A\nabla\phi_\ell \quad \text{in } \Omega_\ell, \\ \nabla \cdot \psi_\ell &= 0 \quad \text{in } \Omega_\ell, \\ \psi_\ell \cdot \mathbf{n} &= \begin{cases} 0 & \text{on } \partial\Omega_\ell \cap \Gamma^*, \\ z & \text{on } \partial\Omega_\ell \cap \gamma, \end{cases} \end{aligned}$$

where $\int_{\partial\Omega_\ell \cap \gamma} z \, ds = 0$. We observe that

$$(2.23) \qquad \int_{\Omega_\ell} P(\lambda)\nabla \cdot \psi_\ell^* \, dx = \int_{\Omega_\ell \cap \gamma} \lambda z \, ds = 0 \,,$$

for every $z \in V_\gamma|_{\partial\Omega_\ell \cap \gamma}$. Thus λ_ℓ is constant on $\gamma \cap \partial\Omega_\ell$. The result that $\lambda_1 = \lambda_2 = \ldots = \lambda_M$ follows by regularity assumption (**R**) and the assumption that Ω is connected. $\Box$

Examples of mixed finite element spaces satisfying the inf-sup convergence condition [1] or the condition that $div(V) \subset W$ can be found in [2, 8, 12, 13]. For these spaces one can establish the following inequality:

$$(2.24) \qquad \sum_i \int_{\partial\Omega_i} A^{-1}\mathbf{U} \cdot \mathbf{U} \, ds \le C_1 h^{-1} \sum_i \int_{\Omega_i} A^{-1}\mathbf{U} \cdot \mathbf{U} \, dx \,,$$

where C_1 is a positive constant and h is a mesh spacing parameter.

Assuming (2.24) we now derive bounds on the bilinear form b.

THEOREM 2.2. *Let* $b : V_\gamma^* \times V_\gamma^* \to \Re$ *be defined by (2.19). Then there exist positive constants* C_0 *and* C_1 *such that*

$$(2.25) \qquad C_0|||\lambda|||^2 \le b(\lambda, \lambda) \le C_1|||\lambda|||^2 h^{-1}$$

where

$$(2.26) \qquad |||\lambda|||^2 = \sum_{i,j} \int_{\gamma_{ij}} \mathbf{n}_{ij}^t A\mathbf{n}_{ij}\lambda^2 \, ds$$

and C_1 *and* h^{-1} *are defined by (2.24).*

Proof. We have

$$\begin{aligned} b(\lambda, \lambda) &= \sum_{ij} \int_{\gamma_{ij}} [\mathbf{U}(\lambda) \cdot \mathbf{n}_{ij}]\lambda \, ds \\ &= \sum_i \int_{\partial\Omega_i \cap \gamma} \lambda\mathbf{U}(\lambda) \cdot \mathbf{n}_i \, ds \\ &= \sum_i \int_{\partial\Omega_i \cap \gamma} \lambda A^{1/2} A^{-1/2}\mathbf{U}(\lambda) \cdot \mathbf{n}_i \, ds \\ &\le \left(\sum_i \int_{\partial\Omega_i \cap \gamma} \lambda^2 \mathbf{n}_i^t A\mathbf{n}_i \, ds \right)^{\frac{1}{2}} \left(\sum_i \int_{\partial\Omega_i \cap \gamma} A^{-1}\mathbf{U}(\lambda) \cdot \mathbf{U}(\lambda) \, ds \right)^{\frac{1}{2}} \\ &\le \sqrt{C_1} h^{-1/2}|||\lambda||| \, \|A^{-1/2}\mathbf{U}(\lambda)\|_{L^2(\Omega)}. \end{aligned}$$

7

Thus,

$$(2.27) \qquad b(\lambda, \lambda) \; = \; \sum_i \int_{\Omega_i} A^{-1} \mathbf{U}(\lambda) \cdot \mathbf{U}(\lambda) \, dx$$

$$(2.28) \qquad\qquad\qquad \leq \; C_1 h^{-1} |||\lambda|||^2 \, .$$

To prove the second half of (2.25), we first consider the case where $\mathbf{meas}(\hat{\Gamma}) > 0$. Let Ω_ℓ be a domain such that $\mathbf{meas}(\partial \Omega_\ell \cap \hat{\Gamma}) > 0$. We set $J_1 = \{\ell\}$. We define $J_k, k = 2, \ldots$ recursively by letting

$$(2.29) \qquad J_k = \{j \mid \mathbf{meas}(\partial \Omega_j \cap \partial \Omega_s) > 0 \text{ for some } s \in J_{k-1}\} \, .$$

We now define a set of auxilary problems. For $i \in J_\ell$, let $\{\psi_i, \phi_i\}$ be the solution to

$$(2.30) \qquad \begin{aligned} \psi_i &= -A \nabla \phi_i \quad \text{in } \Omega_i, \\ \nabla \cdot \psi_i &= 0 \quad \text{in } \Omega_i, \\ \psi_i \cdot \mathbf{n}_i &= \mathbf{n}_i^t A \mathbf{n}_i \lambda \quad \text{on } \gamma_i, \\ \phi_i &= 0 \quad \text{on } \partial \Omega_i \cap \Omega_s, \end{aligned}$$

where $\gamma_i = \partial \Omega_i \setminus \partial \Omega_i \cap \partial \Omega_s$ and $s \in J_{\ell-1}$ with $\mathbf{meas}(\partial \Omega_i \cap \partial \Omega_s) > 0$. Here $\mathbf{n}_i$ denotes the outward normal to Ω_i. We assume $\gamma \subset \bigcup_i \gamma_i$.

Let $\{\psi_i^*, \phi_i^*\} \in V_i \times W_i$ be the mixed finite element approximation to $\{\psi_i, \phi_i\}$ on Ω_i. We have

$$(2.31) \qquad \begin{aligned} \int_{\gamma_i} \mathbf{n}_i^t A \mathbf{n}_i \lambda^2 \, ds &= \int_{\gamma_i} \psi_i \cdot \mathbf{n}_i \lambda \, ds \\ &= \int_{\Omega_i} A^{-1} \mathbf{U}(\lambda) \cdot \psi_i \, dx \\ &= \int_{\Omega_i} A^{-1} \mathbf{U}(\lambda) \cdot (\psi_i^* - \psi_i) \, dx + \int_{\Omega_i} A^{-1} \mathbf{U}(\lambda) \cdot \psi_i \, dx \\ &\leq C \|A^{-1/2} \mathbf{U}(\lambda)\|_{L^2(\Omega_i)} \, \|\phi_i\|_{H^{1+\epsilon}(\Omega_i)} \\ &\leq C \|A^{-1/2} \mathbf{U}(\lambda)\|_{L^2(\Omega_i)} \left(\int_{\gamma_i} \mathbf{n}_i^t A \mathbf{n}_i \lambda^2 \, ds \right)^{\frac{1}{2}} . \end{aligned}$$

Thus,

$$(2.32) \qquad |||\lambda|||^2 \leq \sum_i^M \int_{\gamma_i} \mathbf{n}_i^t A \mathbf{n}_i \lambda^2 \, ds \leq C \, b(\lambda, \lambda),$$

and the result follows.

Estimates for Neumann boundary conditions involve minor modifications to the above argument. Letting $J_1 = \{1\}$ we modify the definition of V_γ^* by

$$(2.33) \qquad V_\gamma^* = \left\{ z \in V_\gamma \mid \int_{\partial \Omega_1 \cup \gamma} \mathbf{n}_1^t A \mathbf{n}_1 z \, ds = 0 \right\} .$$

8

Thus on Ω_1 we can define a local auxiliary problem and bounds are estimated as before. $\square$

As a result of the above theorems and an application of theorems due to Douglas and Douglas [4, 5], we conclude that the multilevel V-cycle method described in detail in [9] for the interface problem (2.19) is convergent.

3. Numerical Experiments. In this section we discuss two numerical experiments that demonstrate the effectiveness of multilevel acceleration, and the efficiency of the resulting algorithm. We begin by defining the Raviart-Thomas spaces. Let Ω be the rectangular domain $(0,1) \times (0,1)$ and

$$\Delta_x : 0 = x_0 < x_1 < \ldots < x_{N_x} = 1,$$
$$\Delta_y : 0 = y_0 < y_1 < \ldots < y_{N_y} = 1,$$

be partitions of $[0,1]$. For such a partition Δ, define

$$M_q^r(\Delta) = \{v \in C^q([0,1]) : v \text{ is a polynomial of degree } \leq r$$
$$\text{on each subinterval of } \Delta\}.$$

When $q = -1$ this is taken to be a space of discontinuous piecewise polynomial functions. Let $\hat{\Gamma}$ be the portion of the boundary of Ω on which Dirichlet boundary conditions are imposed, and let $\Gamma^* = \partial\Omega \setminus \hat{\Gamma}$. The Raviart-Thomas spaces are defined as follows:

$$W^{q,r} = M_q^r(\Delta_x) \otimes M_q^r(\Delta_y),$$
$$\tilde{V}^{q,r} = [M_{q+1}^{r+1}(\Delta_x) \otimes M_q^r(\Delta_y)] \times [M_q^r(\Delta_x) \otimes M_{q+1}^{r+1}(\Delta_y)],$$
$$V^{q,r} = \tilde{V}^{q,r} \cap \{\mathbf{v} : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma^*\}.$$

We use the Douglas and Roberts [8] extension of the Raviart–Thomas spaces for more general domains.

For the purpose of our numerical experiments we will restrict our attention to the Raviart-Thomas space of next to the lowest order ($q = -1, r = 1$) and domains composed of the union of rectangles. The solutions were obtained by applying multigrid V-cycles to the interface problem resulting from (2.19). Two conjugate residual iterations equipped with the inner product

$$(3.1) \qquad (\lambda, \mu) = \sum_{ij} \int_{\gamma_{ij}} \mathbf{n}_{ij}^t A \mathbf{n}_{ij} \lambda \mu \, ds,$$

were used as the smoother. This inner product induces the $||| \cdot |||$–norm defined in (2.26). The resulting subdomain problems were solved to machine precision, a restriction that could perhaps be relaxed. Iteration on the interface problem was carried out

9

until a reduction of six orders of magnitude in the relative residual was achieved. A more complete description of the algorithm may be found in [9]. All calculations were done on Intel iPSC/860 machines located at Oak Ridge National Labs and the Rice Center for Research on Parallel Computation.

In our first experiment we consider potential flow in the "T-shaped" region Ω with boundary Γ depicted in Figure 1. The flow satisfies

$$
\begin{aligned}
\mathbf{u} &= -\nabla p \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \Gamma_{\text{pressure}}, \\
\mathbf{u} \cdot \mathbf{n} &= \begin{cases} 0 & \text{on } \Gamma_{\text{wall}}, \\ -1 & \text{on } \Gamma_{\text{in}}. \end{cases}
\end{aligned}
$$

(3.2)

The region is partitioned into 16 subdomains, each assigned to a separate processor. The velocities for the solution of the test problem are shown in Figure 2. The resulting interface problem is solved using multigrid V-cycles with various numbers of levels. Solution of the interface problem via conjugate gradients equipped with the inner product (3.1) corresponds to the number of levels equal to one. The number of conjugate gradient iterations in this case were 76 for the smaller problem and 108 for the larger one. Table 1 summarizes our results.

Number of Levels	Nx = 48 Time (sec)	Number of V-cycles	Nx = 96 Time (sec)	Number of V-cycles
1	24	-	160	-
2	18	4	96	4
3	11	4	53	4
4	-	-	55	4

TABLE 1

Execution times versus number of levels in V-cycle for "T-shaped" region

From this experiment we see that given a sufficient number of levels, the multilevel scheme improves run time by a factor of two to three over the solution by conjugate gradients applied the interface problem. A portion of the run time comes from computing the exact solution on the coarsest grid. By increasing the number of levels, we decrease the size of that problem and see a corresponding decrease in the run time provided that we do not increase the number of V-cycles required for convergence. We note that the number of V-cycles are independent of the problem size in this example. The next experiment will further demonstrate that this is characteristic of our method of solution.

For the second test problem we consider the solution of a variable coefficient

elliptic Dirichlet problem posed on the unit square S. The problem is

$$
\begin{aligned}
\mathbf{u} &= -a\nabla p \quad \text{in } S, \\
\nabla \cdot \mathbf{u} &= f \quad \text{in } S, \\
p &= 0 \quad \text{on } \partial S.
\end{aligned}
$$

(3.3)

In the above problem f is chosen such that the solution is

$$
u(x,y) = \sum_{n=1,3,5} \left(sin(\pi nx)sin(\pi ny) - \frac{4}{(n\pi)^2} \right).
$$

We consider three functions for $a(x,y)$:

$$
\begin{aligned}
a(x,y) &= 1, \\
a(x,y) &= \frac{1}{1 + 100(x^2 + y^2)}, \\
a(x,y) &= \begin{cases} 100 & \text{if } x < 0.5 \\ 1 & \text{if } x > 0.5. \end{cases}
\end{aligned}
$$

For each choice of a, experiments were run varying spatial discretization and the decomposition into subdomains to determine the efficiency of this algorithm. For a given decomposition, the number of levels used in the V-cycle was chosen such that the smallest dimension in a subdomain on the coarsest grid was 2 grid blocks. Tables 2–4 summarize our results. For the 1x1 decomposition, the number of conjugate gradient iterations are reported.

Discre-tization	Decomposition						
	1x1	1x2	2x2	2x4	4x4	4x8	8x8
16x16	1.7(27)	10.5(3)	5.4(3)	4.2(4)	1.5(3)	0.9(1)	0.5(1)
32x32	7.4(31)	58.6(4)	28.9(4)	14.8(4)	7.4(4)	5.2(4)	3.0(4)
64x64	36.5(41)	292.8(4)	130.6(4)	62.7(4)	29.3(4)	15.8(4)	8.8(4)
128x128	235.6(56)[1]	-	672.6(4)	299.9(4)	127.3(4)	59.2(4)	30.9(4)

TABLE 2

Execution times and Number of V-cycles for $a(x,y) = 1$

Two types of considerations effect the efficiency of Method 2. The primary considerations involve the extra work required to solve (2.5)–(2.7) by iteration on an interface problem (2.19) as compared to a method that is direct in the sense of not introducing such a series of subdomain problems. The secondary considerations are classical issues of processor utilization, communication overhead, and duplicated work. The 1x1

---

[1] Execution time estimated based on comparison times for runs on sequential machine

	Decomposition						
Discre-tization	1x1	1x2	2x2	2x4	4x4	4x8	8x8
16x16	2.0(24)	9.8(3)	5.2(3)	4.3(4)	2.7(4)	1.5(1)	1.4(1)
32x32	8.5(28)	52.4(4)	26.1(4)	14.3(4)	8.0(4)	7.2(4)	6.3(4)
64x64	36.5(39)	252.5(4)	116.7(4)	57.5(4)	29.2(4)	17.3(4)	11.9(4)
128x128	237.4(56)[1]	-	594.3(4)	275.5(4)	126.1(4)	60.1(4)	33.2(4)

TABLE 3

Execution times and Number of V-cycles for $a(x,y) = 1/(1 + 100(x^2 + y^2))$

	Decomposition						
Discre-tization	1x1	1x2	2x2	2x4	4x4	4x8	8x8
16x16	2.0(34)	12.6(3)	8.2(3)	5.0(4)	2.6(4)	1.1(1)	0.8(1)
32x32	8.6(39)	70.8(4)	34.4(4)	17.6(4)	9.1(4)	7.0(4)	5.0(4)
64x64	42.2(50)	347.4(4)	156.5(4)	72.2(4)	35.3(4)	19.5(4)	11.7(4)
128x128	249.4(70)[1]	-	808.0(4)	352.6(4)	155.9(4)	72.1(4)	37.4(4)

TABLE 4

Execution times and Number of V-cycles for $a(x,y) = 100$ if $x > 0.5$, 1 if $x > 0.5$

decomposition corresponds to solving the global problem without decomposition. By comparing the amount of work required to solve this problem to the work required to solve the other decompositions, we determine the efficiency of Method 2.

Processor utilization measurements show that the parallel implementation of this algorithm is dominated by computation for all decompositions where the subdomains are larger than 2x2 grid blocks. Further, the code has minimal duplication of work amongst processors. Hence, issues of efficiency and speedup are actually issues of how the multilevel algorithm for the solution of (2.19) behaves as a function of decomposition and problem size. For a parallel algorithm dominated by computation with no duplicated work, the amount of equivalent sequential work W_{seq} is approximately

$$(3.4) \qquad\qquad W_{seq} = NT_N.$$

where N is the number of processors and T_N is the execution time using N processors. Using this measure of total work, we observe that solving (3.3) by means of the interface problem (2.19) increases the total amount of work by about a factor of 10. For this reason, small decompositions run slower than the sequential 1x1 decomposition. It is important to note that the algorithm does change substantially with decomposition and problem size. For example, the number of interfacial unknowns changes as well as

the number of levels in the V-cycle. In particular, if the subdomains are too small, then acceleration due to a multilevel scheme is limited as shown in the first test problem. The changes in the algorithm also explain the superlinear speedups obtained in the test problem with a discontinuous coefficient.

Several speedup curves have been suggested for evaluating the scaling properties of a parallel algorithm. See for example Worley[15] and the references contained therein. The classical speedup S of an algorithm is

$$(3.5) \qquad\qquad S(N) = \frac{T_1}{T_N}.$$

The curve resulting from plotting speedup versus the number of processors for a problem of fixed size is the classical speedup curve. Gustafson, et al. [11] propose examining a scaled speedup curve where the problem size increases with the number of processors. In view of the comments made in the introduction, both speedup curves are of significance. Figures 3 and 4 display both speedup curves for (3.3).

Another important observation from Tables 2–4 is the independence in the number of V-cycles for a wide range of discretizations and variable coefficients. This independence is a key reason why the algorithm is very scalable. As subdomain problems become too small, however, the amount of exploitable multilevel speedup decreases, and less significantly, the compute to communication ratio decreases. The experiments suggest that for our choice of finite elements and method of solution that there is both sufficient computational work and multilevel acceleration to provide almost perfect linear speedup if the subdomains (and hence the number of processors) are chosen such that they contain at least 16x16 grid blocks, a minimal restriction in view of the comments made in the introduction.

4. Conclusions. We have defined and analyzed a method for solving elliptic problems via domain decomposition and incorporating multilevel acceleration. A proof of the condition number of the resulting interface problem was given. Numerical experiments on an Intel iPSC/860 have shown the resulting algorithm to be scalable due to an insensitivity in the number of V-cycles required for convergence to problem size and variation in coefficients. Since the need to solve larger problems is an important issue in many applications including reservior modeling, the algorithm is particularly promising as demonstrated in the speedup curves. Furthermore, we are encouraged to proceed to a Method 2 implementation in 3-D.

13

Division of Oak Ridge National Labs, the Rice University Center for Research on Parallel Computation, and John Killough at the Houston Petroleum Research Center for their assistance in providing the Intel hypercubes used in this research.

REFERENCES

[1] I. BABUSKA AND J. OSBORN, *Generalized finite element methods: Their performance and their relation to mixed methods*, SIAM J. Numer. Anal., 20 (1983), pp. 510–536.

[2] F. BREZZI, J. DOUGLAS, AND L. MARINI, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.

[3] B. DARLOW, R. EWING, AND M. WHEELER, *Mixed finite element methods for miscible displacement problems in porous media*, in Proceedings of the 6th SPE Symposium on Reservoir Simulation, 1982, pp. 137–146.

[4] C. DOUGLAS, *Multi-grid algorithms for elliptic boundary-value problems*, PhD thesis, Yale University, May 1982.

[5] C. DOUGLAS AND J. DOUGLAS, *Abstract multilevel convergence theory requires almost no assumptions.* to appear, 1990.

[6] J. DOUGLAS, R. EWING, AND M. WHEELER, *The approximation of the pressure by a mixed method in the simulation of miscible displacement*, R.A.I.R.O. Analyse Numerique, 17 (1983), pp. 17–33.

[7] ——, *Time stepping procedures for simulation of miscible displacement using mixed methods for pressure approximation*, R.A.I.R.O. Analyse Numerique, 20 (1986), pp. 77–110.

[8] J. DOUGLAS AND J. ROBERTS, *Global estimates for mixed methods for second order elliptic equations*, Math. of Comp., 44 (1985), pp. 39–52.

[9] R. GLOWINSKI, W. KINTON, AND M. WHEELER, *Acceleration of domain decomposition algorithms for mixed finite elements by multi-level methods*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, ed., Philadelphia, 1990, SIAM, pp. 263–290.

[10] R. GLOWINSKI AND M. F. WHEELER, *Domain decomposition and mixed finite element methods for elliptic problems*, in Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. Golub, G. Meurant, and J. Periaux, eds., Philadelphia, 1988, SIAM, pp. 144–172.

[11] J. GUSTAFSON, G. MONTRY, AND R. BENNER, *Development of parallel methos for a 1024-processor hypercube*, SIAM J. of Sci. Stat. Comput., 9 (1988), pp. 609–638.

[12] J. NEDELEC, *Mixed finite elements in $\Re^3$*, Numer. Math., 35 (1980), pp. 315–341.

[13] P. RAVIART AND J. THOMAS, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical Aspects of the Finite Element Method, no. 606 in Springer Lecture Notes in Mathematics, Springer, Heidelberg, 1977.

[14] T. RUSSELL AND M. WHEELER, *Finite element and finite difference methods for continuous flows in porous media*, in Mathematics of Reservoir Simulation, R. Ewing, ed., SIAM, Philadelphia, 1983, pp. 35–106.

[15] P. WORLEY, *The effect of time constraints on scaled speedup*, SIAM J. of Sci. Stat. Comput., 11 (1990), pp. 838–858.
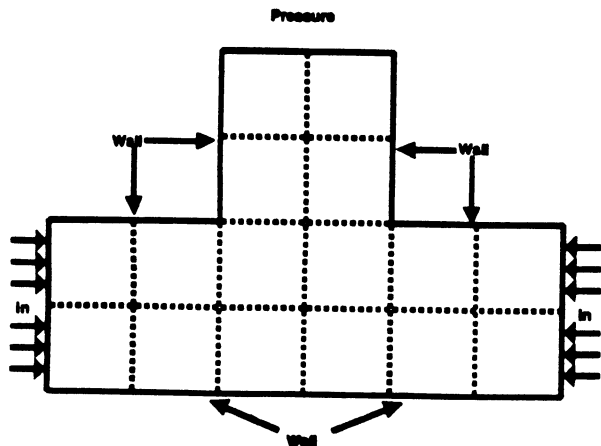
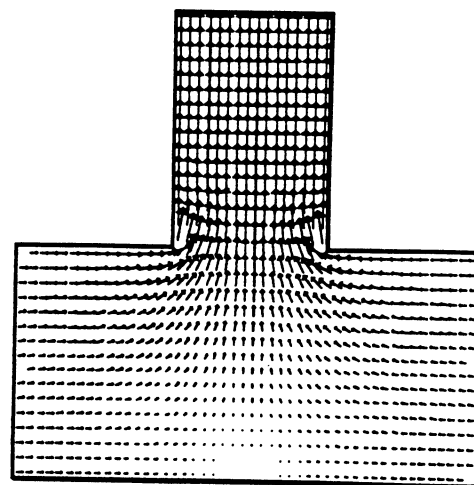FIG. 1. *Schematic of "T-shaped" domain*
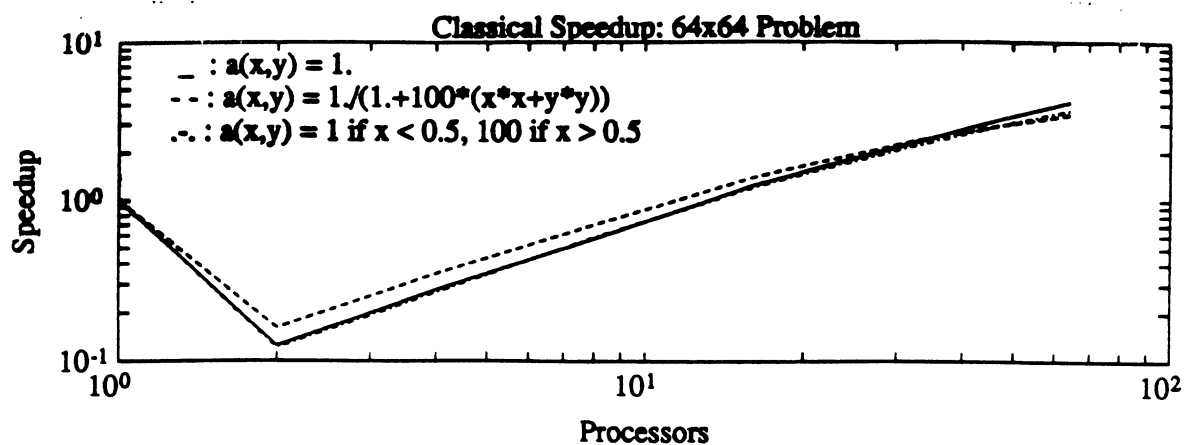

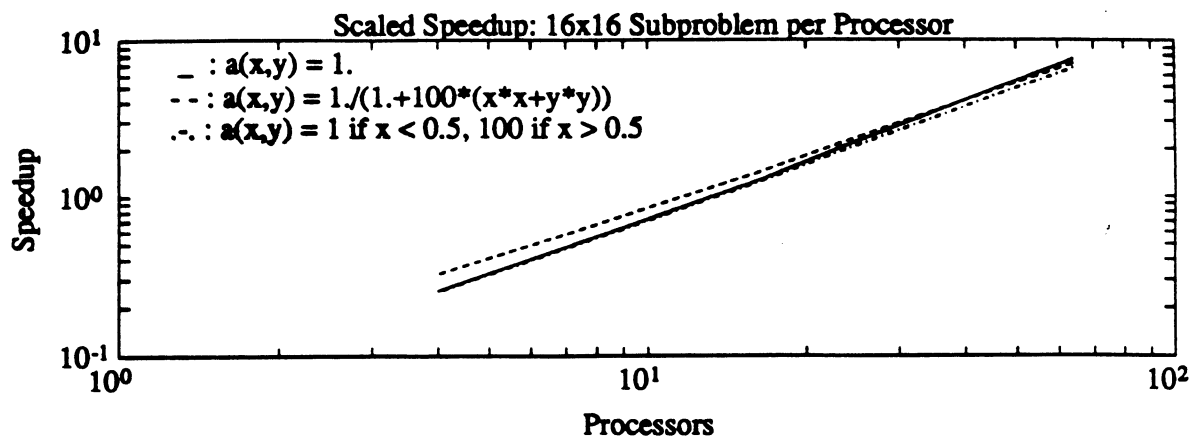FIG. 2. *Velocity profile of first test problem*

FIG. 3. *Classical speedup*

FIG. 4. *Scaled speedup*