A Finite Element Domain Decomposition Method for Parabolic Equations

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Abstract: A domain decomposition procedure for solving parabolic equations in one and two space dimensions is presented, extending earlier work by the authors and T. F. Dupont [1]. The underlying discretization is Galerkin finite elements. In this procedure, subdomain interface data are updated using an explicit procedure in one dimension, and an "implicit in y, explicit in x" procedure in two dimensions. This leads to a time step constraint involving the interface discretization parameter. We concentrate on continuous, piecewise linear and bilinear approximating spaces, and in two dimensions restrict our attention to rectangular elements. The subdomains are nonoverlapping intervals in one dimension, and nonoverlapping strips in two dimensions. *A priori* error estimates in a semi-discrete $L^\infty(L^2)$ norm are derived, which demonstrate that the error is of higher order in the interface discretization parameter, which diminishes the severity of the time step constraint.

Keywords: Domain decomposition, parabolic equations, finite elements, parallel computing

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1 Introduction

With the advent of parallel computers, that is, computers with many processors working simultaneously, substantial increases in computational efficiency in the numerical solution of partial differential equations can, in principle, be obtained by applying algorithms which exploit multi-processor capability. A natural way to solve problems in parallel is to employ domain decomposition. In such an approach, one divides the domain over which the problem is defined into subdomains, and then solves subdomain problems simultaneously. The procedures studied here involve defining values on the subdomain boundaries, and using these values to calculate subdomain solutions, which, when pieced together, provide a reasonable approximation to the true solution.

In [1], we have analyzed a finite difference method which utilizes domain decomposition to solve the heat equation. Here, we intend to demonstrate that similar procedures can be applied to finite element approximation of more general parabolic equations. Our attention is focused on using continuous, piecewise linear approximating spaces and nonoverlapping intervals in one dimension, and rectangular mesh,

tensor product of continuous piecewise linear approximating spaces, and nonoverlapping strips in two space dimensions.

The rest of the paper is organized as follows. In Section 2, we establish some notation. In Section 3, we describe the finite element domain decomposition algorithm for the one dimensional heat equation, and derive an error estimate in a semi-discrete $L^\infty(L^2)$ norm. We verify that, similar to the discussion in [1] on the finite difference approximation, the error introduced by approximating the solution at the subdomain interfaces enters the estimates as a higher-order term, and thus does not substantially affect the overall rate of convergence. In Section 4, we describe a generalization of this method to treat equations with variable coefficients in two space dimensions. Finally, in Section 5, we present numerical results obtained on a multi-processor, shared memory computer.

2 Definition of Norms

Let Ω denote a spatial domain in $\mathbf{R}^d$, $d = 1$ or 2. Denote by $H^m(\Omega)$ and $W^m_\infty(\Omega)$ the standard Sobolev spaces on Ω, with norms $||\cdot||_m$ and $||\cdot||_{\infty,m}$, respectively. Let $L^p(\Omega)$, $p = 2, \infty$, denote the standard Banach spaces, with $||\cdot||$ denoting the L^2 norm, $||\cdot||_\infty$ the L^∞ norm. Finally, denote by $(\cdot,\cdot)$ the L^2 inner product on Ω.

Let $[a,b] \subset [0,T]$ denote a time interval, $X = X(\Omega)$ a Banach or Sobolev space. To incorporate time dependence, we use the notation $L^p(a,b;X)$ and $||\cdot||_{L^p(a,b;X)}$ to denote the space and norm, respectively, of X-valued functions f with the map $t \mapsto ||f(\cdot,t)||_X$ belonging to $L^p(a,b)$. If $[a,b] = [0,T]$, we simplify our notation and write $L^p(X)$ for $L^p(0,T;X)$.

Finally, let $\Delta t = T/M$ for some positive integer M, $t^n = n\Delta t$, $n = 0,\ldots,M$, and $f^n = f(t^n)$.

3 One-space-dimensional Domain Decomposition

Let $u(x,t)$ be the solution of the heat equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in (0,1),\ t \in (0,T], \tag{3.1}$$

$$u(x,0) = u^0(x), \qquad x \in (0,1), \tag{3.2}$$

$$u(0,t) = u(1,t) = 0, \qquad t \in (0,T]. \tag{3.3}$$

We now consider a finite element, domain decomposition algorithm for the approximation of u.

3.1 Basic scheme for two subdomains

We first consider dividing $(0,1)$ into two subdomains, $(0,\bar{x})$ and $(\bar{x},1)$.

Let

$$\delta : 0 = x_0 < x_1 < \ldots < x_{N+1} = 1 \tag{3.4}$$

be a partition of $(0,1)$ into intervals of length $h_i = x_{i+1} - x_i$, $i = 0, \ldots, N$. Assume that $\bar{x} = x_k$ for some k, $1 \leq k \leq N$. Related to $\bar{x}$ we define a parameter $H > 0$ satisfying $H \leq \min(\bar{x}, 1 - \bar{x})$, and we assume $\bar{x} - H$ and $\bar{x} + H$ are points of the partition δ. Let Δt, t^n, and M be defined as in Section 2. For functions f defined at (x,t) for all x and t, let $f_i^n = f(x_i, t^n)$, and let

$$\partial_t f^n = \frac{f^n - f^{n-1}}{\Delta t}.$$

Let $\mathcal{M} = \mathcal{M}(\delta) \subset H^1(0,1)$ denote the space of functions which are continuous on $[0,1]$, linear on each interval (x_i, x_{i+1}), and satisfy the boundary condition at $x = 0$ and $x = 1$. A basis for this space is the set of "hat" functions $\{v_1, \ldots, v_N\}$, given by

$$v_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \leq x \leq x_i, \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x \leq x_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

Define spaces $\mathcal{M}_L$, $\mathcal{M}_R$, and $\mathcal{M}_I$, corresponding to the left and right subdomains and the interface, as follows:

$$\mathcal{M}_L = \{v \in \mathcal{M} |\ v(x) = 0 \text{ for } x \geq \bar{x}\}, \tag{3.6}$$

$$\mathcal{M}_R = \{v \in \mathcal{M} |\ v(x) = 0 \text{ for } x \leq \bar{x}\}, \tag{3.7}$$

and

$$\mathcal{M}_I = \{v \in \mathcal{M} |\ v(x) = 0 \text{ for } x \leq x_{k-1} \text{ and } x \geq x_{k+1}\}. \tag{3.8}$$

Note that $\mathcal{M} = \mathcal{M}_L \oplus \mathcal{M}_R \oplus \mathcal{M}_I$, is the direct sum of these three spaces; for any function $W \in \mathcal{M}$, we have $W = W_L + W_R + W_I$, where

$$W_L(x) = \sum_{i=1}^{k-1} W_i v_i(x) \in \mathcal{M}_L, \tag{3.9}$$

$$W_R(x) = \sum_{i=k+1}^{N} W_i v_i(x) \in \mathcal{M}_R, \tag{3.10}$$

and

$$W_I(x) = W_k v_k(x) \in \mathcal{M}_I. \tag{3.11}$$

We also define an "interface function" $w_I \in \mathcal{M}$ by

$$w_I(x) = \begin{cases} \frac{x - (\bar{x} - H)}{H}, & x \in (\bar{x} - H, \bar{x}], \\ \frac{\bar{x} + H - x}{H}, & x \in (\bar{x}, \bar{x} + H), \\ 0, & \text{otherwise,} \end{cases} \tag{3.12}$$

and define $\mathcal{M}_I^c \subset \mathcal{M}$ by

$$\mathcal{M}_I^c = \{w|\ \ w = \alpha w_I(x) \text{ for some } \alpha \in \mathbf{R}\}. \tag{3.13}$$

For any function g defined at $\bar{x}$, let $\tilde{g} \in \mathcal{M}_I^c$ be given by

$$\tilde{g}(x) = g(\bar{x})w_I(x). \tag{3.14}$$

Our domain decomposition approximation U^n to $u^n = u(t^n)$ is a function in $\mathcal{M}$ determined by the following procedure. First, we approximate the initial condition $u^0(x)$ by

$$U^0 = \pi u^0, \tag{3.15}$$

where $\pi u(t) \in \mathcal{M}$ is the elliptic projection of u, defined by

$$((\pi u(\cdot,t) - u(\cdot,t))_x, v_x) = 0, \quad v \in \mathcal{M}, \quad t \in [0,T]. \tag{3.16}$$

In this simple case, πu interpolates u at the knots $x_j, j = 0, ..., N+1$. Given $U^n(x)$, we first calculate the interface value $U_k^{n+1} = U^{n+1}(\bar{x})$ by

$$\langle \partial_t \tilde{U}^{n+1}\ ,\ w\rangle + (U_x^n, w_x) = 0, \quad w \in \mathcal{M}_I^c, \tag{3.17}$$

where $\langle \cdot\ ,\ \cdot \rangle$ is the trapezoidal rule, or "mass lumped," approximation to $(\cdot,\cdot)$; that is, for $\tilde{g}$ and w in $\mathcal{M}_I^c$,

$$\langle \tilde{g}\ ,\ w\rangle = \tilde{g}_k w_k H. \tag{3.18}$$

Note that (3.17) determines U_k^{n+1} by forward differencing in time, thus $U_I^{n+1}(x)$ is known.

Next, we determine U_L^{n+1} and U_R^{n+1} by

$$(\partial_t U_L^{n+1}, v) + ((U_L^{n+1})_x, v_x) = -(\partial_t U_I^{n+1}, v) - ((U_I^{n+1})_x, v_x), \quad v \in \mathcal{M}_L, \tag{3.19}$$

and

$$(\partial_t U_R^{n+1}, v) + ((U_R^{n+1})_x, v_x) = -(\partial_t U_I^{n+1}, v) - ((U_I^{n+1})_x, v_x), \quad v \in \mathcal{M}_R. \tag{3.20}$$

Note that (3.19) and (3.20) decouple, and can be solved in parallel. Finally,

$$U^{n+1}(x) = U_L^{n+1}(x) + U_R^{n+1}(x) + U_I^{n+1}(x). \tag{3.21}$$

Since (3.17) is explicit in time, one might expect a stability time step constraint involving H. In fact, a stability analysis reveals that the constraint

$$\Delta t \leq H^2/2 \tag{3.22}$$

must hold. However, no constraint involving the h_i's is necessary.

We now prove an L^2 error estimate for the procedure (3.16)-(3.20).

Theorem 1 *Let Δt and H satisfy (3.22), and assume the exact solution u is such that $||u_t||_{L^2(W^2_\infty)}$ and $||u_{tt}||_{L^2(L^\infty)}$ are bounded. Then*

$$\max_{0\le n\le M} ||u^n - U^n|| \le C(\Delta t + h^2 + H^3), \tag{3.23}$$

where $h = \max_i h_i$, and C is a constant which depends on the smoothness of u but not on h, H, or Δt.

Proof of Theorem 1: We use a variation of the standard argument first given in [5]. We compare our approximate solution U with the elliptic projection $\pi u \in \mathcal{M}$ given by (3.16), and then apply the triangle inequality. Below, C will represent a generic constant, independent of h and Δt, and ϵ will denote a generic small constant, also independent of h and Δt.

The following are well known approximation results for $u - \pi u$ [2]:

$$||(\pi u - u)(\cdot,t)|| \le C||u(\cdot,t)||_2 h^2, \tag{3.24}$$

$$||(\pi u_t - u_t)(\cdot,t)|| \le C||u_t(\cdot,t)||_2 h^2. \tag{3.25}$$

Moreover, as remarked earlier, at a point $x_j \in \delta$,

$$(\pi u - u)(x_j,t) = 0, \tag{3.26}$$

and

$$(\pi u_t - u_t)(x_j,t) = 0. \tag{3.27}$$

We note that, by (3.16), and integration by parts in (3.1),

$$(\pi u_x(\cdot,t), v_x) = -(u_t(\cdot,t), v). \tag{3.28}$$

Let $\xi = U - \pi u$, and let $\tilde{\pi u}^{n+1}$ be defined as in (3.14). Then $\xi^0 = 0$, and for $n = 0,\dots,M-1$, $\tilde{\xi}^{n+1} = \tilde{U}^{n+1} - \tilde{\pi u}^{n+1}$ satisfies

$$\langle \partial_t \tilde{\xi}^{n+1}\ ,\ w\rangle + (\xi^n_x, w_x) = -\langle \partial_t \tilde{\pi u}^{n+1}\ ,\ w\rangle + (u^n_t, w), \quad w \in \mathcal{M}^c_I, \tag{3.29}$$

by (3.17) and (3.28). Adding (3.19) and (3.20) and applying (3.28), we obtain

$$(\partial_t \xi^{n+1}, v) + (\xi^{n+1}_x, v_x) = -(\partial_t \pi u^{n+1} - u^{n+1}_t, v), \quad v \in \mathcal{M}_L \oplus \mathcal{M}_R. \tag{3.30}$$

Adding (3.29) to (3.30), with $w = \partial_t \tilde{\xi}^{n+1}$ and $v = \partial_t(\xi^{n+1} - \tilde{\xi}^{n+1}) \equiv \partial_t \bar{\xi}^{n+1}$, we find

$$\begin{aligned}
&||\partial_t \tilde{\xi}^{n+1}||^2_H + ||\partial_t \xi^{n+1}||^2 + \frac{1}{2}\left[\Delta t||\partial_t \xi^{n+1}_x||^2 + \partial_t(||\xi^{n+1}_x||^2)\right] \\
&\quad = (\partial_t \xi^{n+1}, \partial_t \tilde{\xi}^{n+1}) + \Delta t(\partial_t \xi^{n+1}_x, \partial_t \tilde{\xi}^{n+1}_x) + \left[(u^n_t, \partial_t \tilde{\xi}^{n+1}) - \langle \partial_t \tilde{u}^{n+1}\ ,\ \partial_t \tilde{\xi}^{n+1}\rangle\right] \\
&\qquad + (\partial_t(u^{n+1} - \pi u^{n+1}), \partial_t \bar{\xi}^{n+1}) + (u^{n+1}_t - \partial_t u^{n+1}, \partial_t \bar{\xi}^{n+1}) \\
&\quad \equiv I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned} \tag{3.31}$$

where

$$||\tilde{g}||_H^2 \equiv \langle \tilde{g}\ ,\ \tilde{g} \rangle, \quad \tilde{g} \in \mathcal{M}_I^c. \tag{3.32}$$

Note that

$$||\tilde{g}||^2 = \frac{2}{3}||\tilde{g}||_H^2, \tag{3.33}$$

and

$$||\tilde{g}_x||^2 = \frac{2}{H^2}||\tilde{g}||_H^2. \tag{3.34}$$

We now estimate the terms $I_1, \ldots, I_5$. In the arguments to follow we repeatedly use the Cauchy-Schwarz inequality,

$$(f,g) \leq ||f||\ ||g||, \quad f,g, \in L^2,$$

and the inequality,

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2, \quad a,b,\epsilon \in \mathbf{R}, \quad \epsilon > 0, \tag{3.35}$$

or, equivalently,

$$ab \leq \beta a^2 + \frac{1}{4\beta}b^2, \quad a,b,\beta \in \mathbf{R}, \quad \beta > 0. \tag{3.36}$$

We also use (3.33) extensively.

First, choosing $\epsilon = 3/4$ in (3.35), we obtain

$$I_1 \leq \frac{1}{4}||\partial_t \tilde{\xi}^{n+1}||_H^2 + \frac{2}{3}||\partial_t \xi^{n+1}||^2. \tag{3.37}$$

By (3.34) and (3.22),

$$I_2 \leq \frac{2\Delta t}{5}||\partial_t \xi_x^{n+1}||^2 + \frac{5}{8}||\partial_t \tilde{\xi}^{n+1}||_H^2. \tag{3.38}$$

Next, adding and subtracting terms, we have

$$\begin{aligned} I_3 &= (u_t^n - \partial_t u^{n+1}, \partial_t \tilde{\xi}^{n+1}) - \langle \partial_t \tilde{u}^{n+1}\ ,\ \partial_t \tilde{\xi}^{n+1} \rangle + (\partial_t u^{n+1}, \partial_t \tilde{\xi}^{n+1}) \\ &\leq C\Delta t^{-1} \int_{\bar{x}-H}^{\bar{x}+H} \int_{t^n}^{t^{n+1}} |u_t(t,x) - u_t^n(x)|^2 dt dx + \frac{\epsilon}{2}||\partial_t \tilde{\xi}^{n+1}||_H^2 \\ &\quad - \langle \partial_t \tilde{u}^{n+1}\ ,\ \partial_t \tilde{\xi}^{n+1} \rangle + (\partial_t u^{n+1}, \partial_t \tilde{\xi}^{n+1}). \end{aligned} \tag{3.39}$$

By time truncation error analysis, the first term on the right side of (3.39) is bounded by

$$CH\Delta t||u_{tt}||^2_{L^2(t^n,t^{n+1};L^\infty)}. \tag{3.40}$$

Moreover, expanding in a Taylor series, using the fact that the interval over which w_I is nonzero is symmetric about $\bar{x}$, we see that

$$|\langle \partial_t \tilde{u}^{n+1} \, , \, \partial_t \tilde{\xi}^{n+1} \rangle - (\partial_t u^{n+1}, \partial_t \tilde{\xi}^{n+1})| \\ \leq CH^5 \Delta t^{-1} \int_{t^n}^{t^{n+1}} ||u_{txx}(\cdot, t)||_\infty^2 dt + \frac{\epsilon}{2} ||\partial_t \tilde{\xi}^{n+1}||_H^2. \tag{3.41}$$

Thus

$$I_3 \leq CH\Delta t ||u_{tt}||^2_{L^2(t^n, t^{n+1}; L^\infty)} + CH^5 \Delta t^{-1} ||u_{txx}||^2_{L^2(t^n, t^{n+1}; L^\infty)} + \epsilon ||\partial_t \tilde{\xi}^{n+1}||_H^2. \tag{3.42}$$

By (3.25),

$$\begin{aligned} I_4 &\leq C\Delta t^{-1} \int_{t^n}^{t^{n+1}} ||(\pi u^{n+1} - u^{n+1})_t||^2 + \frac{\epsilon}{2} ||\partial_t \bar{\xi}^{n+1}||^2 \\ &\leq C\Delta t^{-1} h^4 ||u_t||^2_{L^2(t^n, t^{n+1}; H^2)} + \epsilon ||\partial_t \tilde{\xi}^{n+1}||_H^2 + \epsilon ||\partial_t \xi^{n+1}||^2, \end{aligned} \tag{3.43}$$

and by backward difference error analysis,

$$I_5 \leq C\Delta t ||u_{tt}||^2_{L^2(t^n, t^{n+1}; L^2)} + \epsilon ||\partial_t \tilde{\xi}^{n+1}||_H^2 + \epsilon ||\partial_t \xi^{n+1}||^2. \tag{3.44}$$

Bounding the right side of (3.31) by (3.37)-(3.44), choosing ϵ sufficiently small ($\epsilon < 1/24$ is sufficient) and hiding the appropriate terms on the left, multiplying by Δt, and summing on n, $n = 0, \ldots, m-1 < M$, we obtain

$$\sum_{n=0}^{m-1} \left[||\partial_t \tilde{\xi}^{n+1}||_H^2 + ||\partial_t \xi^{n+1}||^2 + \Delta t ||\partial_t \xi_x^{n+1}||^2 \right] \Delta t + ||\xi_x^m||^2 \\ \leq Ch^4 ||u_t||^2_{L^2(H^2)} + CH^5 ||u_t||^2_{L^2(W^2_\infty)} \\ + CH\Delta t^2 ||u_{tt}||^2_{L^2(L^\infty)} + C\Delta t^2 ||u_{tt}||^2_{L^2(L^2)}. \tag{3.45}$$

Note that, at this point, since m is arbitrary, we have an $\mathcal{O}(h^2 + \Delta t + H^{\frac{5}{2}})$ estimate for $||\xi_x^m||$. Thus, an estimate for $||U^m - u^m||$ has been obtained, at a loss of $H^{\frac{1}{2}}$. In order to obtain an H^3 estimate, we proceed.

Setting $w = \tilde{\xi}^{n+1}$ in (3.29) and $v = \bar{\xi}^{n+1} = \xi^{n+1} - \tilde{\xi}^{n+1}$ in (3.30), and adding the resulting equations, we obtain

$$\begin{aligned} &\frac{1}{2} \partial_t \left(||\tilde{\xi}^{n+1}||_H^2 + ||\xi^{n+1}||^2 \right) + \frac{\Delta t}{2} \left(||\partial_t \tilde{\xi}^{n+1}||_H^2 + ||\partial_t \xi^{n+1}||^2 \right) + ||\xi_x^{n+1}||^2 \\ &\quad = (\partial_t \xi^{n+1}, \tilde{\xi}^{n+1}) + \Delta t (\partial_t \xi_x^{n+1}, \tilde{\xi}_x^{n+1}) + \left[(u_t^n, \tilde{\xi}^{n+1}) - \langle \partial_t \tilde{u}^{n+1} \, , \, \tilde{\xi}^{n+1} \rangle \right] \\ &\qquad + (\partial_t u^{n+1} - \partial_t \pi u^{n+1}, \bar{\xi}^{n+1}) + (u_t^{n+1} - \partial_t u^{n+1}, \bar{\xi}^{n+1}) \\ &\quad \equiv I_6 + I_7 + I_8 + I_9 + I_{10}. \end{aligned} \tag{3.46}$$

Before estimating $I_6 - I_{10}$, we note that there exists a constant C_I, independent of h, such that

$$|\xi_i^{n+1}|^2 \leq C_I ||\xi_x^{n+1}||^2, \quad 1 \leq i \leq N. \tag{3.47}$$

(Actually, $C_I = 1/4$.)

Using (3.47), we find that

$$\begin{aligned} I_6 &\leq CC_I H||\partial_t \xi^{n+1}||^2 + \frac{\epsilon}{C_I}|\xi_k^{n+1}|^2 \\ &\leq CH||\partial_t \xi^{n+1}||^2 + \epsilon||\xi_x^{n+1}||^2. \end{aligned} \tag{3.48}$$

Similarly, we have

$$\begin{aligned} I_7 &\leq CC_I H\Delta t||\partial_t \xi_x^{n+1}||^2 + \frac{\epsilon\Delta t}{C_I H}||\tilde{\xi}_x^{n+1}||^2 \\ &= CH\Delta t||\partial_t \xi_x^{n+1}||^2 + \frac{2\epsilon\Delta t}{C_I H^2}|\xi_k^{n+1}|^2 \\ &\leq CH\Delta t||\partial_t \xi_x^{n+1}||^2 + \epsilon||\xi_x^{n+1}||^2. \end{aligned} \tag{3.49}$$

Here we have used (3.34), (3.22), and (3.47).

For the term I_8, we must bound

$$(u_t^n - \partial_t u^{n+1}, \tilde{\xi}^{n+1}) + \left[(\partial_t u^{n+1}, \tilde{\xi}^{n+1}) - \langle \partial_t \tilde{u}^{n+1}\ ,\ \tilde{\xi}^{n+1}\rangle\right].$$

Following the argument used to derive (3.42) and applying (3.47) we find that

$$I_8 \leq CH^2\Delta t||u_{tt}||^2_{L^2(t^n,t^{n+1};L^\infty)} + CH^6\Delta t^{-1}||u_t||^2_{L^2(t^n,t^{n+1};W^2_\infty)} + \epsilon||\xi_x^{n+1}||^2. \tag{3.50}$$

Moreover,

$$\begin{aligned} I_9 &= (\partial_t(u^{n+1} - \pi u^{n+1}), \xi^{n+1}) - (\partial_t(u^{n+1} - \pi u^{n+1}), \tilde{\xi}^{n+1}) \\ &\leq CC_I\Delta t^{-1}h^4(1+H)||u_t||^2_{L^2(t^n,t^{n+1};H^2)} + \frac{\epsilon}{C_I}(||\xi^{n+1}||^2 + |\xi_k^{n+1}|^2) \\ &\leq CC_I\Delta t^{-1}h^4(1+H)||u_t||^2_{L^2(t^n,t^{n+1};H^2)} + \epsilon||\xi_x^{n+1}||^2, \end{aligned} \tag{3.51}$$

and

$$I_{10} \leq C\Delta t(1+H)||u_{tt}||^2_{L^2(t^n,t^{n+1};L^2)} + \epsilon||\xi_x^{n+1}||^2. \tag{3.52}$$

Substituting (3.48)-(3.52) into (3.46), choosing $\epsilon < \frac{1}{5}$, multiplying by Δt, and summing on n, we obtain

$$LHS \leq C(\Delta t^2 + h^4 + H^6) + CH\sum_{n=0}^{m-1}\left[||\partial_t \xi^{n+1}||^2\Delta t + ||\partial_t \xi_x^{n+1}||^2\Delta t^2\right] \tag{3.53}$$

where

$$LHS = ||\tilde{\xi}^m||_H^2 + ||\xi^m||^2 + \sum_{n=0}^{m-1}||\xi_x^{n+1}||^2\Delta t. \tag{3.54}$$

Applying (3.45) to the last term on the right of (3.53), we obtain

$$||\xi^m||^2 \leq C(\Delta t^2 + h^4 + H^6). \tag{3.55}$$

Since m is arbitrary, then by the triangle inequality, (3.55), and (3.24), we complete the proof of the theorem. //

3.2 Extension to many subdomains

In this section, we generalize the scheme presented above to the case of more than two subdomains. Let

$$0 < H_1 \leq \bar{x}_1 < \ldots < \bar{x}_K \leq 1 - H_K < 1 \tag{3.56}$$

denote interface points between subdomains, with related parameters $H_l > 0$, $l = 1, \ldots, K$. We assume $\bar{x}_l$, $\bar{x}_l - H_l$, and $\bar{x}_l + H_l$ are points of δ for each l. We also assume that

$$\bar{x}_{l-1} \leq \bar{x}_l - H_l, \tag{3.57}$$

and

$$\bar{x}_{l-1} + H_{l-1} \leq \bar{x}_l, \quad l = 2, \ldots, K. \tag{3.58}$$

We decompose $\mathcal{M}$ as

$$\mathcal{M} = \left(\oplus_{l=1}^{K} \mathcal{M}_{I,l}\right) \oplus \left(\mathcal{M}/(\oplus_{l=1}^{K} \mathcal{M}_{I,l})\right),$$

where $\mathcal{M}_{I,l}$ is the analogue of $\mathcal{M}_I$ for $\bar{x}_l$. We also define interface functions $w_{I,l}$ by

$$w_{I,l}(x) = \begin{cases} \frac{x-(\bar{x}_l - H_l)}{H_l}, & \bar{x}_l - H_l \leq x \leq \bar{x}_l, \\ \frac{\bar{x}_l + H_l - x}{H_l}, & \bar{x}_l < x \leq \bar{x}_l + H_l, \\ 0, & \text{otherwise}, \end{cases} \tag{3.59}$$

and denote by $\mathcal{M}_{I,l}^c$ the analogue of $\mathcal{M}_I^c$ for $\bar{x}_l$. Moreover, for g defined at $\bar{x}_l$, let $\tilde{g}_l \in \mathcal{M}_{I,l}^c$ be given by

$$\tilde{g}_l(x) = g(\bar{x}_l) w_{I,l}(x). \tag{3.60}$$

In this case,

$$||\tilde{g}_l||_{H_l}^2 = \langle \tilde{g}_l \, , \, \tilde{g}_l \rangle = |\tilde{g}_l(\bar{x}_l)|^2 H_l = |g(\bar{x}_l)|^2 H_l. \tag{3.61}$$

Set $U^0 = \pi u^0$ as before. For each interface, let $\tilde{U}_l^{n+1} \in \mathcal{M}_{I,l}^c$. Then $\tilde{U}_l^{n+1}(\bar{x}_l) = U^{n+1}(\bar{x}_l)$, $l = 1, \ldots, K$, is found by

$$\langle \partial_t \tilde{U}_l^{n+1} \, , \, w \rangle + (U_x^n, w_x) = 0, \quad w \in \mathcal{M}_{I,l}^c, \quad l = 1, \ldots, K. \tag{3.62}$$

These equations determine $U_{I,l}^{n+1} \in \mathcal{M}_{I,l}$, which are the analogues of U_I^{n+1} above. Note that they can be solved simultaneously. Next, we solve,

$$(\partial_t U^{n+1}, v) + (U_x^{n+1}, v_x) = 0, \quad v \in \mathcal{M}/\left(\oplus_{l=1}^{K} \mathcal{M}_{I,l}\right), \tag{3.63}$$

which decomposes into subdomain problems which can also be solved simultaneously.

We note that, under the assumptions (3.57) and (3.58),

$$||\sum_{l=1}^{K} \tilde{g}_l||^2 \leq \sum_{l=1}^{K} ||\tilde{g}_l||^2_{H_l}, \tag{3.64}$$

$$||\sum_{l=1}^{K} (\tilde{g}_l)_x||^2 \leq \frac{4}{\bar{H}^2} \sum_{l=1}^{K} ||\tilde{g}_l||^2_{H_l}, \tag{3.65}$$

for $\tilde{g}_l \in \mathcal{M}^c_{I,l}$. We assume Δt and $\bar{H}$ satisfy

$$\frac{\Delta t}{\bar{H}^2} \leq \frac{1}{4}, \tag{3.66}$$

where $\bar{H} = \min_l H_l$. When considering only the L^2-stability of the scheme, it is sufficient that

$$\frac{\Delta t}{\bar{H}^2} \leq \frac{3}{8}.$$

The scheme (3.62)-(3.63) satisfies the following *a priori* error estimate.

Theorem 2 *Assume that u satisfies the smoothness assumptions given in Theorem 1, and (3.66) holds. Then the algorithm (3.62)-(3.63) satisfies*

$$\max_n ||u^n - U^n|| \leq C(\Delta t + h^2 + KH(\Delta t + h^2 + H^2)), \tag{3.67}$$

where $H = \max_l H_l$.

Proof of Theorem 2: The proof follows closely the proof of Theorem 1. The analogue of (3.31) is

$$\begin{aligned}
&\sum_{l=1}^{K} ||\partial_t \tilde{\xi}_l^{n+1}||^2_{H_l} + ||\partial_t \xi^{n+1}||^2 + \frac{1}{2}\left[\Delta t ||\partial_t \xi_x^{n+1}||^2 + \partial_t(||\xi_x^{n+1}||^2)\right] \\
&\quad = \sum_{l=1}^{K} \Big\{(\partial_t \xi^{n+1}, \partial_t \tilde{\xi}_l^{n+1}) + \Delta t(\partial_t \xi_x^{n+1}, \partial_t(\tilde{\xi}_l)_x^{n+1}) \\
&\qquad + \left[(u_t^n, \partial_t \tilde{\xi}_l^{n+1}) - \langle \partial_t \tilde{\pi} u_l^{n+1}\,,\, \partial_t \tilde{\xi}_l^{n+1}\rangle\right] \\
&\qquad -(\partial_t(u^{n+1} - \pi u^{n+1}), \partial_t \tilde{\xi}_l^{n+1}) - (u_t^{n+1} - \partial_t u^{n+1}, \partial_t \tilde{\xi}_l^{n+1})\Big\} \\
&\qquad + (\partial_t(u^{n+1} - \pi u^{n+1}), \partial_t \xi^{n+1}) + (u_t^{n+1} - \partial_t u^{n+1}, \partial_t \xi^{n+1}).
\end{aligned} \tag{3.68}$$

Multiplying (3.68) by Δt, summing on n, and applying the arguments used to bound $I_1 - I_5$ above, with the bounds (3.64)-(3.66) applied where needed, we find that

$$\begin{aligned}
S^* \;=\; &\sum_{n=0}^{m-1} \Delta t \left[\sum_{l=1}^{K} ||\partial_t \tilde{\xi}_l^{n+1}||^2_{H_l} + ||\partial_t \xi^{n+1}||^2\right] \\
&+\frac{1}{2}\left[\sum_{n=0}^{m-1} \Delta t^2 ||\partial_t \xi_x^{n+1}||^2 + ||\xi_x^m||^2\right]
\end{aligned} \tag{3.69}$$

satisfies

$$
\begin{aligned}
S^* \;\le\; & (\alpha+\epsilon)\sum_{n=0}^{m-1}||\partial_t\xi^{n+1}||^2\Delta t+\left(\frac{1}{4\alpha}+\frac{1}{4\beta}+2\epsilon\right)\sum_{n=0}^{m-1}\sum_{l=1}^{K}||\partial_t\tilde{\xi}_l^{n+1}||_{H_l}^2\Delta t \\
& +\beta\sum_{n=0}^{m-1}\Delta t^2||\partial_t\xi_x^{n+1}||^2+C\Delta t^2||u_{tt}||_{L^2(L^2)}^2+Ch^4||u_t||_{L^2(H^2)}^2 \qquad (3.70)\\
& +CKHh^4||u_t||_{L^2(H^2)}^2+CKH\Delta t^2||u_{tt}||_{L^2(L^\infty)}^2+CKH^5||u_{txx}||_{L^2(L^\infty)}^2,
\end{aligned}
$$

where α, ϵ, and β are positive constants. Choosing $\epsilon=\frac{1}{32}$, $\alpha=\frac{4}{5}$, and $\beta=\frac{2}{5}$, then the coefficient of the second term on the right of (3.70) is bounded by one, and we have

$$
\Delta t\sum_{n=0}^{m-1}\left[||\partial_t\xi^{n+1}||^2+\Delta t||\partial_t\xi_x^{n+1}||^2\right]\le C(\Delta t^2+h^4+KH(h^4+\Delta t^2+H^4)). \quad (3.71)
$$

The analogue of (3.46) is

$$
\begin{aligned}
&\frac{1}{2}\partial_t\left(\sum_{l=1}^{K}||\tilde{\xi}_l^{n+1}||_{H_l}^2+||\xi^{n+1}||^2\right)+\frac{\Delta t}{2}\left(\sum_{l=1}^{K}||\partial_t\tilde{\xi}_l^{n+1}||_{H_l}^2+||\partial_t\xi^{n+1}||^2\right)+||\xi_x^{n+1}||^2\\
&=\sum_{l=1}^{K}\Big\{(\partial_t\xi^{n+1},\tilde{\xi}_l^{n+1})+\Delta t(\partial_t\xi_x^{n+1},(\tilde{\xi}_l)_x^{n+1})+\left[(\partial_t u^{n+1},\tilde{\xi}_l^{n+1})-\langle\partial_t(\tilde{u}_l)^{n+1}\,,\,\tilde{\xi}_l^{n+1}\rangle\right]\\
&\qquad+((u)_t^n-\partial_t u^{n+1},\tilde{\xi}_l^{n+1})-(\partial_t(u^{n+1}-\pi u^{n+1}),\tilde{\xi}_l^{n+1})\\
&\qquad-(u_t^{n+1}-\partial_t u^{n+1},\tilde{\xi}_l^{n+1})\Big\}\\
&\quad+(\partial_t(u^{n+1}-\pi u^{n+1}),\xi^{n+1})+(u_t^{n+1}-\partial_t u^{n+1},\xi^{n+1})\\
&\le CKH\left[||\partial_t\xi^{n+1}||^2+\Delta t||\partial_t\xi_x^{n+1}||^2\right]+\frac{\epsilon}{4C_IKH}||\sum_{l=1}^{K}\tilde{\xi}_l^{n+1}||^2+\frac{\epsilon\Delta t}{4C_IKH}||\sum_{l=1}^{K}(\tilde{\xi}_l)_x^{n+1}||^2\\
&\quad+CK^2H^6\Delta t^{-1}||u_t||_{L^2(t^n,t^{n+1};W_\infty^2)}^2+CK^2H^2\Delta t||u_{tt}||_{L^2(t^n,t^{n+1};L^\infty)}^2\\
&\quad+Ch^4\Delta t^{-1}(K^2H^2+KH+1)||u_t||_{L^2(t^n,t^{n+1};H^2)}^2\\
&\quad+C\Delta t(KH+1)||u_{tt}||_{L^2(t^n,t^{n+1};L^2)}^2+\frac{\epsilon}{2C_I}\max_i|\xi_i^{n+1}|^2.\\
&\le CKH||\partial_t\xi^{n+1}||^2+CKH\Delta t||\partial_t\xi_x^{n+1}||^2\\
&\quad+CK^2H^6\Delta t^{-1}||u_t||_{L^2(t^n,t^{n+1};W_\infty^2)}^2+CK^2H^2\Delta t||u_{tt}||_{L^2(t^n,t^{n+1};L^\infty)}^2\\
&\quad+Ch^4\Delta t^{-1}(K^2H^2+KH+1)||u_t||_{L^2(t^n,t^{n+1};H^2)}^2\\
&\quad+C\Delta t(KH+1)||u_{tt}||_{L^2(t^n,t^{n+1};L^2)}^2+\epsilon||\xi_x^{n+1}||^2. \qquad (3.72)
\end{aligned}
$$

Here we have employed arguments similar to those used to bound I_6-I_{10} above, and applied (3.64)-(3.66) and the inequality (3.47). Multiplying above by Δt, summing on n, and applying the estimate (3.71), we find

$$
||\xi^m||^2\le C(\Delta t^2+h^4+K^2H^2(\Delta t^2+h^4+H^4)).
$$

Invoking the triangle inequality and (3.24) completes the proof of Theorem 2. //

4 Two-space-dimensional Domain Decomposition

In this section, let $\Omega = (0,1) \times (0,1)$, and let $u(x,y,t)$ satisfy

$$\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = 0, \quad (x,y) \in \Omega, \quad t \in (0,T], \tag{4.1}$$

$$u(x,y,0) = u^0(x,y), \qquad (x,y) \in \Omega, \tag{4.2}$$

$$u(0,y,t) = u(1,y,t) = 0, \quad y \in (0,1), \quad t \in (0,T], \tag{4.3}$$

$$u(x,0,t) = u(x,1,t) = 0, \quad x \in (0,1), \quad t \in (0,T]. \tag{4.4}$$

Here $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, and $D = (D_{ij}(x,y))$ is a smooth, symmetric, positive definite, two-by-two matrix, satisfying,

$$D_* I < D < D^* I, \tag{4.5}$$

for positive constants D_* and D^*.

4.1 Basic scheme for two subdomains

We consider dividing the domain Ω into two subdomains, $(0,\bar{x}) \times (0,1)$ and $(\bar{x},1) \times (0,1)$.

Denote by

$$\delta_x : 0 = x_0 < x_1 < \ldots < x_{N_x+1} = 1$$

a partition of $(0,1)$ into intervals of length $h_i^x = x_{i+1} - x_i$, $i = 0, \ldots, N_x$. Similarly, denote by δ_y a partition of $(0,1)$ into intervals of length $h_j^y = y_{j+1} - y_j$, $j = 0, \ldots, N_y$, and let $\delta = \delta_x \otimes \delta_y$ define a partition of Ω into rectangles. Let $H > 0$ be defined as in Section 3.1, and assume $\bar{x}$ $(= x_k)$, $\bar{x} - H$, and $\bar{x} + H$ are all points of δ_x.

We let $\mathcal{M} \subset H^1(\Omega)$ denote the space of continuous functions on Ω, bilinear on each rectangle defined by δ, and zero on $\partial\Omega$, and note that a basis for $\mathcal{M}$ is the tensor product $\{v_{\delta_x,1}(x), \ldots, v_{\delta_x,N_x}(x)\} \otimes \{v_{\delta_y,1}(y), \ldots, v_{\delta_y,N_y}(y)\}$ of hat functions defined as in (3.5), with respect to the partitions δ_x and δ_y. We define spaces $\mathcal{M}_L$, $\mathcal{M}_R$, and $\mathcal{M}_I$ analogous to (3.6)-(3.8); for example,

$$\mathcal{M}_L = \{v \in \mathcal{M} | \ v(x,y) = 0 \text{ for } x \geq \bar{x}\}, \tag{4.6}$$

and we write $W \in \mathcal{M}$ as

$$\begin{aligned} W(x) &= \sum_{j=1}^{N_y} \sum_{i=1}^{k-1} W_{ij} v_{\delta_x,i}(x) v_{\delta_y,j}(y) + \sum_{j=1}^{N_y} \sum_{i=k+1}^{N_x} W_{ij} v_{\delta_x,i}(x) v_{\delta_y,j}(y) \\ &\quad + \sum_{j=1}^{N_y} W_{kj} v_{\delta_x,k}(x) v_{\delta_y j}(y) \\ &\equiv W_L(x) + W_R(x) + W_I(x); \end{aligned} \tag{4.7}$$

hence, $W_L \in \mathcal{M}_L$, $W_R \in \mathcal{M}_R$, and $W_I \in \mathcal{M}_I$.

We define interface functions $w_{I,j}$ by

$$w_{I,j}(x,y) = w_I(x)v_{\delta_y,j}(y), \quad j = 1,\ldots,N_y, \tag{4.8}$$

where w_I is given by (3.12), and set

$$\mathcal{M}_I^c = \text{span } \{w_{I,1},\ldots,w_{I,N_y-1}\}. \tag{4.9}$$

For g defined at $(\bar{x},y)$, $y \in (0,1)$, let

$$\tilde{g}(x,y) = g(\bar{x},y)w_I(x). \tag{4.10}$$

Hence, for $W \in \mathcal{M}$, $\tilde{W} = \sum_{j=1}^{N_y} W_{kj}w_I(x)v_{\delta_y,j}(y) \in \mathcal{M}_I^c$. Also note that $\mathcal{M}_I^c \subset \mathcal{M}$.

In this section, the elliptic projection $\pi u \in \mathcal{M}$ is defined by

$$(D\nabla\pi u(\cdot,t),\nabla v) = (D\nabla u(\cdot,t),\nabla v) = -(u_t(\cdot,t),v), \quad v \in \mathcal{M}, \quad t \in [0,T]. \tag{4.11}$$

Our domain decomposition approximation $U^n \in \mathcal{M}$ to u^n is given by the following:

$$U^0 = \pi u^0. \tag{4.12}$$

For $n = 0,\ldots,M-1$,

$$\langle \partial_t \tilde{U}^{n+1}\ ,\ w\rangle + (D\nabla U^n,\nabla w) + \Delta t(D_{22}\partial_t \tilde{U}_y^{n+1}, w_y) = 0, \quad w \in \mathcal{M}_I^c, \tag{4.13}$$

where, in this case, $\langle\cdot\ ,\ \cdot\rangle$ is an approximation to $(\cdot,\cdot)$ using the trapezoidal rule in x, that is,

$$\langle \partial_t \tilde{U}^{n+1}\ ,\ w\rangle = H\int_0^1 \partial_t U^{n+1}(\bar{x},y)w(\bar{x},y)dy. \tag{4.14}$$

Thus, the interface values U_{kj}^{n+1}, $j = 1,\ldots,N_y-1$, are found by solving a tridiagonal system of equations. This determines U_I^{n+1}; U_L^{n+1} and U_R^{n+1} are determined by

$$(\partial_t U_L^{n+1},v) + (D\nabla U_L^{n+1},\nabla v) = -(\partial_t U_I^{n+1},v) - (D\nabla U_I^{n+1},\nabla v), \quad v \in \mathcal{M}_L, \tag{4.15}$$

and

$$(\partial_t U_R^{n+1},v) + (D\nabla U_R^{n+1},\nabla v) = -(\partial_t U_I^{n+1},v) - (D\nabla U_I^{n+1},\nabla v), \quad v \in \mathcal{M}_R. \tag{4.16}$$

As in one space dimension, these equations decouple, and can be solved in parallel. We assume, for concreteness, that Δt and H satisfy

$$\frac{\Delta t}{H^2}||D_{11}||_\infty \le \frac{1}{6}. \tag{4.17}$$

When considering only stability, a more favorable constraint with upper bound of $5/12$ can be assumed.

The algorithm satisfies the following estimate:

Theorem 3 *Assume (4.17) holds, and assume u is smooth such that $||u_t||_{L^2(W^2_\infty)}$, $||u||_{L^\infty(W^2_\infty)}$ and $||u_{tt}||_{L^2(L^\infty)}$ are bounded. Then U given by (4.13)-(4.16) satisfies*

$$\max_n ||u^n - U^n|| \le C(\Delta t + h^2 + Hh^2|lnh| + H^3), \tag{4.18}$$

where $h = \max(\max_i h_i^x, \max_j h_j^y)$, and C depends on the smoothness of u, D_*, and D^*, but not on h, H, or Δt.

Proof of Theorem 3: Again, we compare the discrete solution with the elliptic projection πu. Estimates of the form (3.24) and (3.25) hold. We will also use the following L^∞ estimates, which, for smooth D, are extensions [4] of estimates in [3] for Laplace's equation:

$$||u - \pi u||_{1,\infty} \le Ch||u||_{2,\infty}, \tag{4.19}$$

and

$$||u - \pi u||_\infty \le Ch^2 \ln\frac{1}{h}||u||_{2,\infty}. \tag{4.20}$$

Let $\eta = u - \pi u$, then adding (4.13), (4.15), and (4.16), and applying (4.11), the error $\xi = U - \pi u$ satisfies $\xi^0 = 0$, and for $n = 0, \ldots, M-1$,

$$\begin{aligned}
&\langle \partial_t \tilde{\xi}^{n+1} \ , \ w\rangle + (\partial_t \xi^{n+1}, v+w) + (D\nabla \xi^{n+1}, \nabla(v+w)) + \Delta t(D_{22}\partial_t \tilde{\xi}_y^{n+1}, w_y) \\
&\quad = (D\nabla(\xi^{n+1} - \xi^n), \nabla w) + \langle \partial_t \tilde{\eta}^{n+1} \ , \ w\rangle - (\partial_t u^{n+1} - u_t^n, w) \\
&\quad\quad - \left[\langle \partial_t \tilde{u}^{n+1} \ , \ w\rangle - (\partial_t u^{n+1}, w)\right] + (\partial_t \eta^{n+1}, v) - (\partial_t u^{n+1} - u_t^{n+1}, v) \\
&\quad\quad + (\partial_t \xi^{n+1}, w) - \Delta t(D_{22}\partial_t \tilde{\pi u}_y^{n+1}, w_y),
\end{aligned} \tag{4.21}$$

where $w \in \mathcal{M}_I^c$, and $v \in \mathcal{M}_L \oplus \mathcal{M}_R$.

Similar to the one dimensional case, we again divide the proof into two steps. First, setting $w = \partial_t \tilde{\xi}^{n+1}$ and $v = \partial_t \bar{\xi}^{n+1} \equiv \partial_t(\xi^{n+1} - \tilde{\xi}^{n+1})$ in (4.21), we find

$$S^n = \sum_{l=1}^{8} E_l, \tag{4.22}$$

where

$$\begin{aligned}
S^n &= ||\partial_t \tilde{\xi}^{n+1}||_H^2 + ||\partial_t \xi^{n+1}||^2 + \frac{1}{2}\partial_t ||D^{\frac{1}{2}}\nabla \xi^{n+1}||^2 \\
&\quad + \frac{\Delta t}{2}||D^{\frac{1}{2}}\nabla \partial_t \xi^{n+1}||^2 + \Delta t||D_{22}^{\frac{1}{2}}\partial_t \tilde{\xi}_y^{n+1}||^2.
\end{aligned} \tag{4.23}$$

The terms E_l, $l = 1, \ldots, 8$ will be analyzed one by one. First,

$$\begin{aligned}
E_1 &= (D\nabla(\xi^{n+1} - \xi^n), \nabla \partial_t \tilde{\xi}^{n+1}) \\
&\le \frac{\Delta t}{2}\left(\alpha||D^{\frac{1}{2}}\nabla \partial_t \xi^{n+1}||^2 + \frac{1}{\alpha}||D^{\frac{1}{2}}\nabla \partial_t \tilde{\xi}^{n+1}||^2\right),
\end{aligned} \tag{4.24}$$

where $0 < \alpha < 1$. Consider the last term. Using (3.34), which also holds in this case, and the fact that D is symmetric positive definite,

$$\begin{aligned} ||D^{\frac{1}{2}}\nabla\partial_t\tilde{\xi}^{n+1}||^2 &= ||D_{11}^{\frac{1}{2}}\partial_t\tilde{\xi}_x^{n+1}||^2 + ||D_{22}^{\frac{1}{2}}\partial_t\tilde{\xi}_y^{n+1}||^2 + 2(D_{12}\partial_t\tilde{\xi}_x^{n+1}, \partial_t\tilde{\xi}_y^{n+1}) \\ &\leq (1+\lambda)||D_{11}^{\frac{1}{2}}\partial_t\tilde{\xi}_x^{n+1}||^2 + (1+\frac{1}{\lambda})||D_{22}^{\frac{1}{2}}\partial_t\tilde{\xi}_y^{n+1}||^2 \\ &\leq \frac{2(1+\lambda)}{H^2}||D_{11}||_\infty||\partial_t\tilde{\xi}^{n+1}||_H^2 + (1+\frac{1}{\lambda})||D_{22}^{\frac{1}{2}}\partial_t\tilde{\xi}_y^{n+1}||^2 \quad (4.25) \end{aligned}$$

where $\lambda > 0$. Thus,

$$\begin{aligned} E_1 \leq \frac{\alpha\Delta t}{2}||D^{\frac{1}{2}}\nabla\partial_t\xi^{n+1}||^2 + \frac{\Delta t(1+\lambda)}{\alpha H^2}||D_{11}||_\infty||\partial_t\tilde{\xi}^{n+1}||_H^2 \\ + \frac{\Delta t}{2\alpha}(1+\frac{1}{\lambda})||D_{22}\partial_t\tilde{\xi}_y^{n+1}||^2. \quad (4.26) \end{aligned}$$

Differentiating (4.11) with respect to t, and applying the estimate (4.20) for η_t,

$$\begin{aligned} E_2 &= \langle\partial_t\tilde{\eta}^{n+1}\,,\,\partial_t\tilde{\xi}^{n+1}\rangle \\ &\leq CH\Delta t^{-1}\int_{t^n}^{t^{n+1}}\int_0^1 |u_t(\bar{x},y,t) - \pi u_t(\bar{x},y,t)|^2 dydt + \epsilon||\partial_t\tilde{\xi}^{n+1}||_H^2 \\ &\leq CH\Delta t^{-1}\cdot h^4(\ln h)^2||u_t||^2_{L^2(t^n,t^{n+1};W^2_\infty)} + \epsilon||\partial_t\tilde{\xi}^{n+1}||_H^2. \quad (4.27) \end{aligned}$$

By time truncation error analysis,

$$\begin{aligned} E_3 &= -(\partial_t u^{n+1} - u_t^n, \partial_t\tilde{\xi}^{n+1}) \\ &\leq CH\Delta t||u_{tt}||^2_{L^2(t^n,t^{n+1};L^\infty)} + \epsilon||\partial_t\tilde{\xi}^{n+1}||_H^2. \quad (4.28) \end{aligned}$$

Now consider

$$\begin{aligned} E_4 &= (\partial_t u^{n+1}, \partial_t\tilde{\xi}^{n+1}) - \langle\partial_t\tilde{u}^{n+1}\,,\,\partial_t\tilde{\xi}^{n+1}\rangle \\ &= \int_0^1 \sum_j \partial_t\xi_{kj}^{n+1}v_j(y)\cdot\left[H\partial_t u^n(\bar{x},y) - \int_{\bar{x}-H}^{\bar{x}+H} w_I(x)\partial_t u^n(x,y)dx\right]dy. \end{aligned}$$

Similar to the one-dimensional case, by Taylor's expansion in x,

$$E_4 \leq CH^5\Delta t^{-1}||u_t||^2_{L^2(t^n,t^{n+1};W^2_\infty)} + \epsilon||\partial_t\tilde{\xi}^{n+1}||_H^2. \quad (4.29)$$

Moreover,

$$\begin{aligned} E_5 &= (\partial_t\eta^{n+1}, \partial_t(\xi^{n+1} - \tilde{\xi}^{n+1})) \\ &\leq Ch^4\Delta t^{-1}||u_t||^2_{L^2(t^n,t^{n+1};H^2)} + \epsilon||\partial_t\xi^{n+1}||^2 + \epsilon||\partial_t\tilde{\xi}^{n+1}||_H^2, \quad (4.30) \\ E_6 &= -(\partial_t u^{n+1} - u_t^{n+1}, \partial_t(\xi^{n+1} - \tilde{\xi}^{n+1})) \\ &\leq C\Delta t||u_{tt}||^2_{L^2(t^n,t^{n+1};L^2)} + \epsilon||\partial_t\xi^{n+1}||^2 + \epsilon||\partial_t\tilde{\xi}^{n+1}||_H^2, \quad (4.31) \end{aligned}$$

and

$$\begin{aligned} E_7 &= (\partial_t \xi^{n+1}, \partial_t \tilde{\xi}^{n+1}) \\ &\leq \beta ||\partial_t \xi^{n+1}||^2 + \frac{1}{6\beta} ||\partial_t \tilde{\xi}^{n+1}||_H^2, \end{aligned} \tag{4.32}$$

where $0 < \beta < 1$. Here we have used (3.33), which also holds in this case.

Finally, consider E_8,

$$\begin{aligned} E_8 &= -\Delta t (D_{22} \partial_t \tilde{\pi u}_y^{n+1}, \partial_t \tilde{\xi}_y^{n+1}) \\ &= -\Delta t (D_{22} \partial_t (\tilde{\pi u}_y^{n+1} - \tilde{u}_y^{n+1}), \partial_t \tilde{\xi}_y^{n+1}) \\ &\quad -\Delta t (D_{22} \partial_t \tilde{u}_y^{n+1}, \partial_t \tilde{\xi}_y^{n+1}) \\ &\equiv E_{8,1} + E_{8,2}. \end{aligned} \tag{4.33}$$

By (4.19),

$$\begin{aligned} E_{8,1} &\leq C \int_{t^n}^{t^{n+1}} ||D_{22}^{\frac{1}{2}} ((\tilde{\pi u}_t - \tilde{u}_t)_y)||^2 + \epsilon \Delta t ||D_{22}^{\frac{1}{2}} \partial_t \tilde{\xi}_y^{n+1}||^2 \\ &\leq C H h^2 ||u_t||^2_{L^2(t^n, t^{n+1}; W_\infty^2)} + \epsilon \Delta t ||D_{22}^{\frac{1}{2}} \partial_t \tilde{\xi}_y^{n+1}||^2; \end{aligned} \tag{4.34}$$

and, integrating by parts in y,

$$\begin{aligned} E_{8,2} &= \Delta t ((D_{22} \partial_t \tilde{u}_y^{n+1})_y, \partial_t \tilde{\xi}^{n+1}) \\ &\leq C H \Delta t ||(D_{22} u_{ty})_y||^2_{L^2(t^n, t^{n+1}; L^\infty)} + \epsilon ||\partial_t \tilde{\xi}^{n+1}||_H^2. \end{aligned} \tag{4.35}$$

Substituting (4.26)-(4.35) into (4.23), multiplying by Δt, and summing on n, we obtain

$$\begin{aligned} \sum_{n=0}^{m-1} S^n \Delta t \leq\ & \frac{\alpha \Delta t}{2} \sum_{n=0}^{m-1} \Delta t ||D^{\frac{1}{2}} \nabla \partial_t \xi^{n+1}||^2 + \left(\frac{\Delta t}{2\alpha} (1 + \frac{1}{\lambda}) + \epsilon \Delta t \right) \sum_{n=0}^{m-1} \Delta t ||D_{22}^{\frac{1}{2}} \partial_t \tilde{\xi}_y^{n+1}||^2 \\ & + \left(\frac{\Delta t (1+\lambda)}{\alpha H^2} ||D_{11}||_\infty + \frac{1}{6\beta} + 6\epsilon \right) \sum_{n=0}^{m-1} \Delta t ||\partial_t \tilde{\xi}^{n+1}||_H^2 \\ & + (\beta + 2\epsilon) \sum_{n=0}^{m-1} \Delta t ||\partial_t \xi^{n+1}||^2 + C H^5 ||u_t||^2_{L^2(W_\infty^2)} \\ & + C H h^4 (\ln h)^2 ||u_t||^2_{L^2(W_\infty^2)} + C H \Delta t^2 ||u_{tt}||^2_{L^2(L^\infty)} \\ & + C h^4 ||u_t||^2_{L^2(H^2)} + C \Delta t^2 ||u_{tt}||^2_{L^2(L^2)} \\ & + C H h^2 \Delta t ||u_t||^2_{L^2(W_\infty^2)} + C H \Delta t^2 ||(D_{22} u_{ty})_y||^2_{L^2(L^\infty)}. \end{aligned} \tag{4.36}$$

Choosing $\alpha = \frac{11}{12}$, $\beta = \frac{5}{6}$, $\lambda = \frac{3}{2}$, and $\epsilon < \frac{1}{30}$, and noting that

$$H h^2 \Delta t \leq \frac{H h^4}{2} + \frac{H \Delta t^2}{2},$$

then, under the condition (4.17), we have

$$\hat{S} \leq C(H^5 + h^4 + \Delta t^2 + H (\ln h)^2 h^4), \tag{4.37}$$

where

$$\hat{S} = \Delta t \sum_{n=0}^{m-1} \left[||\partial_t \tilde{\xi}^{n+1}||_H^2 + ||\partial_t \xi^{n+1}||^2 + \Delta t ||D_{22}^{\frac{1}{2}} \partial_t \tilde{\xi}_y^{n+1}||^2 \right] + ||D^{\frac{1}{2}} \nabla \xi^m||^2. \tag{4.38}$$

Setting $w = \tilde{\xi}^{n+1}$ and $v = \xi^{n+1} - \tilde{\xi}^{n+1}$ in equation (4.21), we have

$$\begin{aligned} \tilde{S}^n &= \Delta t (D \nabla \partial_t \xi^{n+1}, \nabla \tilde{\xi}^{n+1}) + \langle \partial_t \tilde{\eta}^{n+1} , \tilde{\xi}^{n+1} \rangle \\ &+ (u_t^n - \partial_t u^{n+1}, \tilde{\xi}^{n+1}) - \left[\langle \partial_t u^{n+1} , \tilde{\xi}^{n+1} \rangle - (\partial_t u^{n+1}, \tilde{\xi}^{n+1}) \right] + (\partial_t \eta^{n+1}, \xi^{n+1} - \tilde{\xi}^{n+1}) \\ &- (\partial_t u^{n+1} - u_t^{n+1}, \xi^{n+1} - \tilde{\xi}^{n+1}) + (\partial_t \xi^{n+1}, \tilde{\xi}^{n+1}) - \Delta t (D_{22} \partial_t \tilde{\pi} u_y^{n+1}, \tilde{\xi}_y^{n+1}) \\ &\equiv \sum_{l=1}^{8} \tilde{E}_l, \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} \tilde{S}^n &= \frac{1}{2} \partial_t \left(||\tilde{\xi}^{n+1}||_H^2 + ||\xi^{n+1}||^2 + \Delta t ||D_{22}^{\frac{1}{2}} \tilde{\xi}_y^{n+1}||^2 \right) + ||D^{\frac{1}{2}} \nabla \xi^{n+1}||^2 \\ &+ \frac{\Delta t}{2} \left(||\partial_t \tilde{\xi}^{n+1}||_H^2 + ||\partial_t \xi^{n+1}||^2 + \Delta t ||D_{22}^{\frac{1}{2}} \partial_t \tilde{\xi}_y^{n+1}||^2 \right). \end{aligned} \tag{4.40}$$

Again, we need the estimates on all the terms $\tilde{E}_1, \ldots, \tilde{E}_8$. Most of the estimates are very close to the estimates for the terms $E_1, \ldots, E_8$ above, except for the terms $\tilde{E}_1, \tilde{E}_8$, where summing by parts is used. Thus, we state some of the results without giving all the details.

Multiplying by Δt, summing on n, summing by parts on n, and following arguments similar to those used for E_1 above,

$$\begin{aligned} \Delta t \sum_{n=0}^{m-1} \tilde{E}_1^n &= -\Delta t \sum_{n=1}^{m-1} \Delta t (D \nabla \xi^n, \nabla \partial_t \tilde{\xi}^{n+1}) + \Delta t (D \nabla \xi^m, \nabla \tilde{\xi}^m) \\ &\leq \frac{\Delta t}{2} \sum_{n=1}^{m-1} ||D^{\frac{1}{2}} \nabla \xi^n||^2 + \frac{\Delta t^2}{2} \sum_{n=1}^{m-1} ||D^{\frac{1}{2}} \nabla \partial_t \tilde{\xi}^{n+1}||^2 \Delta t \\ &+ C \Delta t ||D^{\frac{1}{2}} \nabla \xi^m||^2 + \epsilon \Delta t ||D^{\frac{1}{2}} \nabla \tilde{\xi}^m||^2 \\ &\leq \frac{\Delta t}{2} \sum_{n=1}^{m-1} ||D^{\frac{1}{2}} \nabla \xi^n||^2 + C \Delta t \sum_{n=0}^{m-1} \left[||\partial_t \tilde{\xi}^{n+1}||_H^2 \Delta t + \Delta t^2 ||D_{22}^{\frac{1}{2}} \partial_t \tilde{\xi}_y^{n+1}||^2 \right] \\ &+ C \Delta t ||D^{\frac{1}{2}} \nabla \xi^m||^2 + \epsilon ||\tilde{\xi}^m||_H^2 + \epsilon \Delta t ||D_{22}^{\frac{1}{2}} \tilde{\xi}_y^m||^2. \end{aligned} \tag{4.41}$$

Here we have used the analogue of (4.25) to bound $\Delta t ||D^{\frac{1}{2}} \nabla \tilde{\xi}^m||^2$.

By (4.20),

$$\tilde{E}_2 \leq C H^2 \Delta t^{-1} h^4 (\ln h)^2 ||u_t||_{L^2(t^n, t^{n+1}; W_\infty^2)}^2 + \frac{\epsilon}{H} ||\tilde{\xi}^{n+1}||_H^2. \tag{4.42}$$

Note that by using a one dimensional imbedding result, we have for $v \in \mathcal{M}$,

$$\begin{aligned} \int_0^1 v^2(\bar{x},y)dy &\leq \int_0^1 \left(C \int_0^1 v_x^2(x,y)dx \right) dy \\ &= C||v_x||^2, \end{aligned} \tag{4.43}$$

and, consequently,

$$||\tilde{v}||_H^2 \leq CH||v_x||^2. \tag{4.44}$$

Moreover, since $\mathcal{M} \in H_0^1(\Omega)$,

$$||v||^2 \leq C||\nabla v||^2. \tag{4.45}$$

Thus, by (4.44) and (4.43),

$$\tilde{E}_2 \leq CH^2\Delta t^{-1}h^4(\ln h)^2||u_t||^2_{L^2(t^n,t^{n+1};W^2_\infty)} + \epsilon||\nabla\xi^{n+1}||^2, \tag{4.46}$$

$$\tilde{E}_3 \leq CH^2\Delta t||u_{tt}||^2_{L^2(t^n,t^{n+1};L^\infty)} + \epsilon||\nabla\xi^{n+1}||^2, \tag{4.47}$$

and, by the argument used to bound E_4 above, and (4.43),

$$\tilde{E}_4 \leq CH^6\Delta t^{-1}||u_t||^2_{L^2(t^n,t^{n+1};W^2_\infty)} + \epsilon||\nabla\xi^{n+1}||^2. \tag{4.48}$$

For the terms $\tilde{E}_5$ and $\tilde{E}_6$, we use (4.44) and (4.45) to deduce that

$$\begin{aligned} \tilde{E}_5 + \tilde{E}_6 &\leq C(1+H)h^4\Delta t^{-1}||u_t||^2_{L^2(t^n,t^{n+1};H^2)} \\ &\quad + C(1+H)\Delta t||u_{tt}||^2_{L^2(t^n,t^{n+1};L^2)} + \epsilon||\nabla\xi^{n+1}||^2. \end{aligned} \tag{4.49}$$

Furthermore, by (4.43),

$$\tilde{E}_7 \leq CH||\partial_t\xi^{n+1}||^2 + \epsilon||\nabla\xi^{n+1}||^2. \tag{4.50}$$

Finally, adding and substracting $(D_{22}\partial_t\tilde{u}_y^n, \tilde{\xi}_y^{n+1})$, multiplying by Δt, summing on n, summing by parts on n, and integrating $(D_{22}\tilde{u}_y^n, \partial_t\tilde{\xi}_y^{n+1})$ and $(D_{22}\tilde{u}_y^m, \tilde{\xi}_y^m)$ by parts in y, we obtain

$$\begin{aligned} \Delta t \sum_{n=0}^{m-1} \tilde{E}_8^n &= \Delta t \sum_{n=0}^{m-1} \Delta t(D_{22}(\tilde{\eta}_y^n - \tilde{u}_y^n), \partial_t\tilde{\xi}_y^{n+1}) \\ &\quad + \Delta t(D_{22}(\tilde{\eta}_y^m - \tilde{u}_y^m), \tilde{\xi}_y^m) \\ &\leq C(\Delta t h^2 + \Delta t^2) \sum_{n=0}^{m-1} ||u^n||^2_{2,\infty}\Delta t \\ &\quad + C(\Delta t h^2 H + \Delta t^2)||u||^2_{L^\infty(W^2_\infty)} + \epsilon\Delta t||D_{22}^{\frac{1}{2}}\tilde{\xi}_y^m||^2 + \epsilon||\tilde{\xi}^m||_H^2 \\ &\quad + CH \sum_{n=0}^{m-1} \left[||\partial_t\tilde{\xi}^{n+1}||_H^2\Delta t + \Delta t^2||D_{22}^{\frac{1}{2}}\partial_t\tilde{\xi}_y^{n+1}||^2 \right]. \end{aligned} \tag{4.51}$$

Choosing $\epsilon = \epsilon(D_*)$ sufficiently small, summing (4.39) on n, substituting the above estimates on $\tilde{E}_1$ through $\tilde{E}_8$, and hiding appropriate terms, we obtain

$$\begin{aligned}||\xi^m||^2 \;\leq\; & C(\Delta t + H)\sum_{n=0}^{m-1}\left[||\partial_t\tilde{\xi}^{n+1}||_H^2\Delta t + \Delta t^2||D_{22}^{\frac{1}{2}}\partial_t\tilde{\xi}_y^{n+1}||^2\right] \\ & + C\Delta t||D^{\frac{1}{2}}\nabla\xi^m||^2 + CH\sum_{n=0}^{m-1}||\partial_t\xi^{n+1}||^2\Delta t + CH^2(\ln h)^2h^4||u_t||^2_{L^2(W^2_\infty)} \\ & + CH^2\Delta t^2||u_{tt}||^2_{L^2(L^\infty)} + CH^6||u_t||^2_{L^2(W^2_\infty)} + Ch^4||u_t||^2_{L^2(H^2)} \\ & + C\Delta t^2||u_{tt}||^2_{L^2(L^2)} + C(\Delta t^2 + h^4)||u||^2_{L^\infty(W^2_\infty)}. \end{aligned} \tag{4.52}$$

Finally, applying (4.37), we find that

$$||\xi^m||^2 \leq C(\Delta t^2 + h^4 + H^6 + H^2(\ln h)^2h^4). \tag{4.53}$$

By applying the triangle inequality and (3.24), this completes the proof of (4.18). //

4.2 Many subdomains

An extension to a multiple strip decomposition is now straightforward. In particular, we assume a decomposition into $K+1$ subdomains $(\bar{x}_l, \bar{x}_{l+1}) \times (0,1)$, $l = 0, \ldots, K$, with $\bar{x}_0 = 0$, $\bar{x}_{K+1} = 1$, and $\bar{x}_l$, $l = 1, \ldots, K$, interior interface points with parameters $H_l > 0$, each satisfying the constraint on H given above. As before, we assume $\bar{x}_l - H_l$, $\bar{x}_l$, and $\bar{x}_l + H_l$ are points of δ_x, and we assume (3.57) and (3.58) hold. At each interior interface, we solve explicit equations of the form (4.13). These equations decouple and can be solved in parallel, as can the interior equations, which are of the form of (4.15).

We have the following result.

Theorem 4 *Assume that u satisfies the smoothness assumptions in Theorem 3, and*

$$||D_{11}||_\infty \frac{\Delta t}{\bar{H}^2} < \frac{1}{8}, \tag{4.54}$$

where $\bar{H} = \min_l H_l$. *Then, the multidomain algorithm satisfies*

$$\max_n ||u^n - U^n|| \;\leq\; C(\Delta t + h^2 + KH(\Delta t + h^2 + h^2|lnh| + H^2)), \tag{4.55}$$

where $H = \max_l H_l$.

The proof of Theorem 4 is similar to the proof of Theorem 3, just as the proof of Theorem 2 was similar to the proof of Theorem 1, and is omitted.

When considering only L^2-stability, we require that

$$\frac{\Delta t}{\bar{H}^2}||D_{11}||_\infty < \frac{5}{24}.$$

Remark: The estimate (4.20), which is used in the proofs of Theorems 3 and 4, is a "worst-case" approximation to the actual error at the interface. The resulting term in (4.55) is of order $KHh^2|\ln h|$. The constant K is related to the number of subdomains, and in practice is bounded by the number of processors. The parameter H goes to zero with h and Δt, thus, under reasonable assumptions on K and H, $KH|\ln h|$ also goes to zero with h, or is at least bounded as $h \to 0$.

5 Numerical results

We conclude by presenting some numerical results for the two-dimensional scheme analyzed above. We consider two problems on $\Omega = [0,1] \times [0,1]$:

Problem 1:

$$u_t - \Delta u = 0, \quad (x,y) \in \Omega, \quad t \in (0,T], \tag{5.56}$$

$$u(0,y,t) = 1, \quad u(1,y,t) = 0, \quad y \in (0,1), \quad t \in (0,T], \tag{5.57}$$

$$u(x,0,t) = u(x,1,t) = 0, \quad x \in (0,1), \quad t \in (0,T], \tag{5.58}$$

$$u(x,y,0) = 0, \quad (x,y) \in \Omega. \tag{5.59}$$

Problem 2:

$$u_t - \Delta u = f(x,y), \quad (x,y) \in \Omega, \quad t \in (0,T], \tag{5.60}$$

$$u(x,y,t) = t, \quad (x,y) \in \partial\Omega, \tag{5.61}$$

$$u(x,y,0) = 16x(1-x)y(1-y), \quad (x,y) \in \Omega, \tag{5.62}$$

where $f(x,y) = 1 + 32(x(1-x) + y(1-y))$, which has solution $u(x,y,t) = t + 16x(1-x)y(1-y)$.

We compare our scheme for 1, 2, 4, and 8 subdomains. These runs were performed on an Alliant FX/8 computer, at the Advanced Research Computing Facility, Argonne National Laboratory. This computer has a shared memory architecture with 8 processors. The scheme for 1 subdomain was equivalent to the standard backward-in-time Galerkin method. The discrete system of equations generated by the method was solved using preconditioned conjugate gradient iteration, with diagonal preconditioning. Therefore, the timings of the runs are effected not only by parallelization, but by the number of iterations required to converge the conjugate gradient routine in each subdomain. Hence, it is possible to obtain speed-up by more than the expected factor when subdividing the problem.

In these runs, a global 80 by 80 uniform mesh was employed. The time step was .001. In the decomposition, $\bar{x}_l = l/(K+1)$, $l = 1, \ldots, K$, where $K+1$ is the number of subdomains. Timings for the scheme as the number of subdomains varied are presented in Tables 1 and 2. The times reported were averaged over several runs performed at 0% capacity (we were the only users); 20 time steps were taken. We also report the number of conjugate gradient iterations, averaged over time and the number of subdomains, that is, if q_l^n represents the number of conjugate gradient iterations required in subdomain l at time t^n, we compute

$$\bar{q} = \frac{1}{M(K+1)} \sum_{n=1}^{M} \sum_{l=1}^{K+1} q_l^n,$$

where M is the total number of time steps ($M = 20$ in this case). This number should in general decrease as the number of subdomains increases, since the number

Problem 1		
# of s.d.	CPU time (sec)	aver. no. of c.g. iter.
1	71.1	23
2	44.5	23
4	22.2	17
8	11.2	10

Table 1: Timings for Problem 1

Problem 2		
# of s.d.	CPU time (sec)	aver. no. of c.g. iter.
1	87.5	28
2	43.5	26
4	23.0	24
8	17.6	19

Table 2: Timings for Problem 2

of unknowns per subdomain decreases as we subdivide, and since the number of iterations for convergence using preconditioned conjugate gradient is dependent on the number of unknowns.

By examining Tables 1 and 2, we see that the algorithm performs quite well, especially in the 1-4 processor range, and does well even for 8 processors for Problem 1. These results are not meant to be conclusive, and the performance of the scheme will certainly vary from one machine to another.

Solutions for some of these runs are plotted in Figures 1 and 2. In Figure 1, we compare the fully implicit solution with the domain decomposition solution with 8 subdomains, at t = .02 and y = 1/2, for Problem 1. In Figure 2, we compare the true solution with the domain decomposition solution with 4 subdomains at time t = .1 and y = 1/2 for Problem 2.

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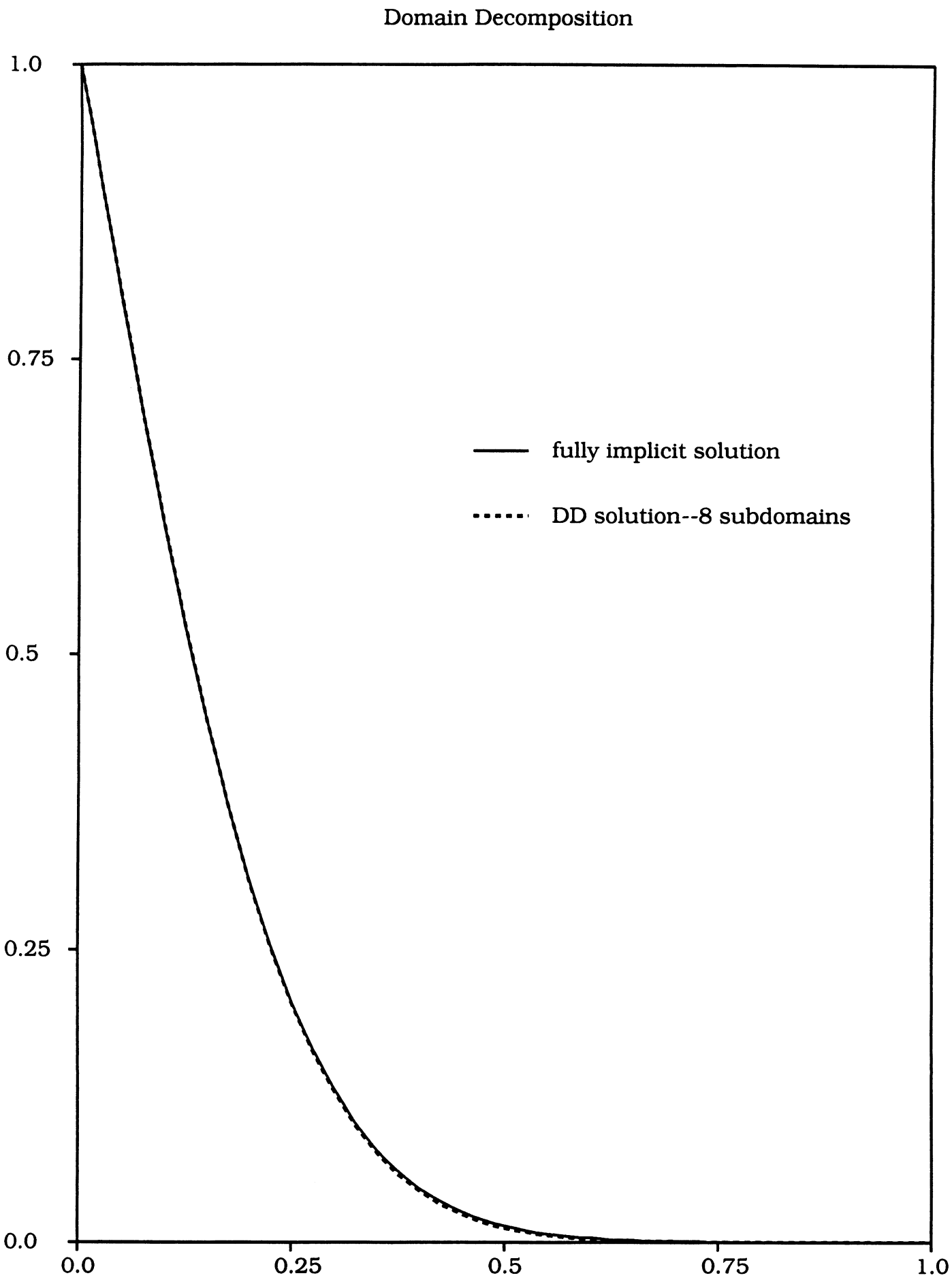

Figure 1: DD solution vs. fully implicit solution at t = .02

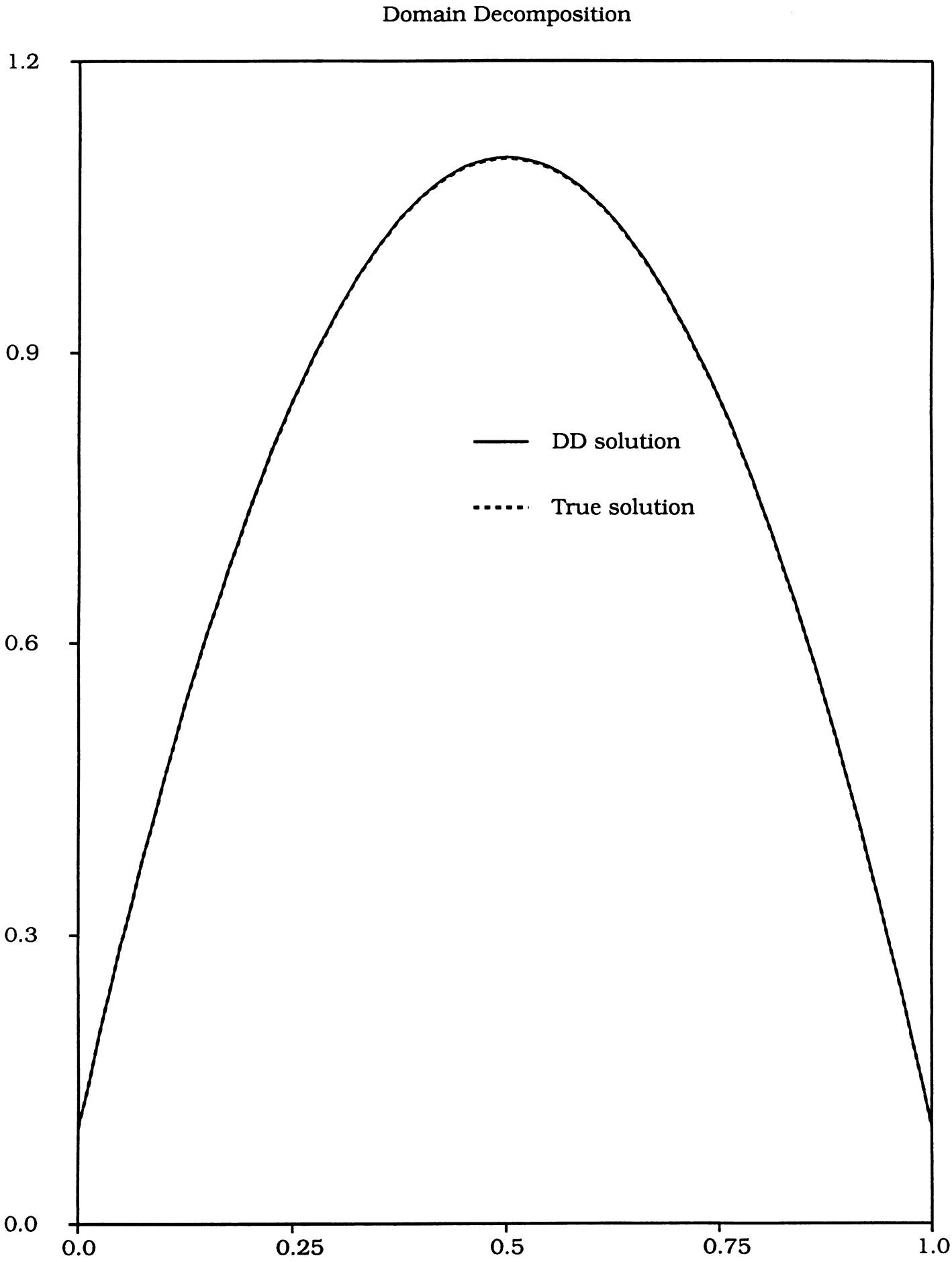

Figure 2: DD solution vs. true solution at t = .1