Mortar Finite Difference (Finite volume) Methods for Multiphase Flow in Multiblock Reservoirs

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Objectives and Motivation for Numerical Method

Develop a discretization method that:

- is accurate and conserves mass locally
- can handle tensor permeabilities and dispersivities
- can handle irregularly shaped domains
- can be implemented as cell-centered finite differencees
- can be incorporated into existing simulators using rectangular grids

Logically rectangular multiblock griding strategy

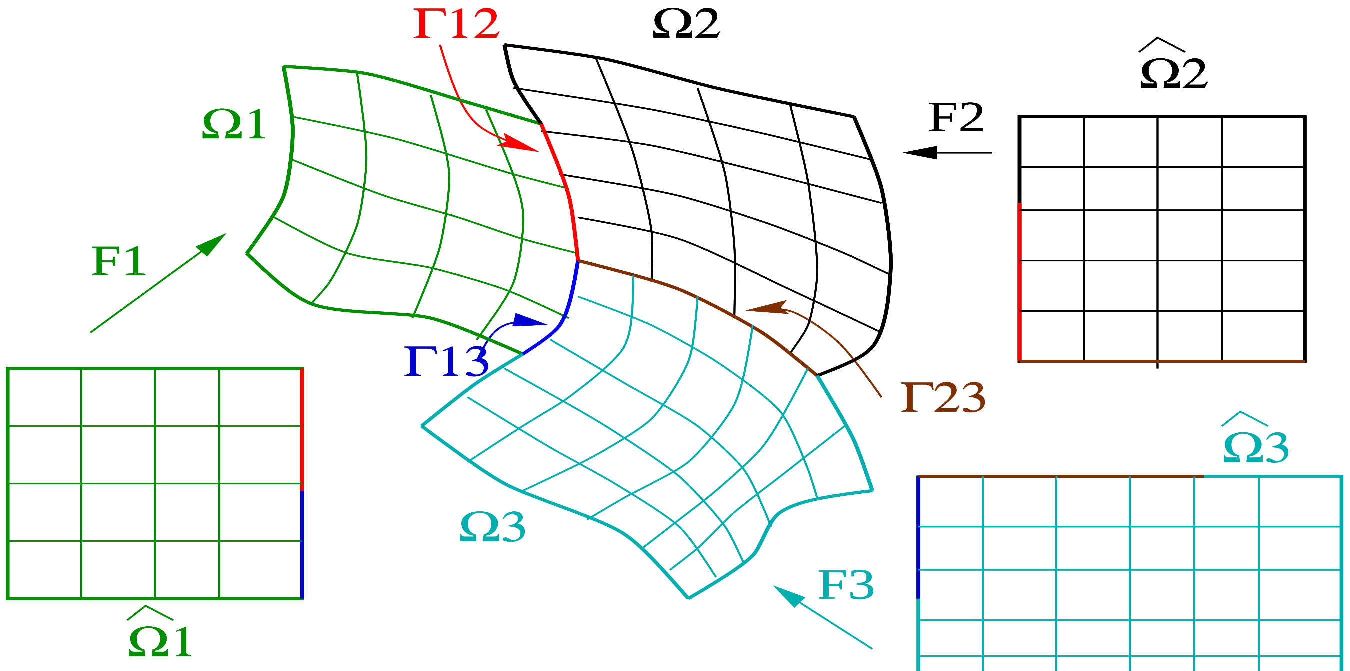
Partition the domain into a series of blocks:

$$\bar{\Omega} = \cup_{i=1}^{n} \bar{\Omega}_i, \quad \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$$

Consider each block as the image of a rectangular reference block:

$$F_i : \hat{\Omega}_i \to \Omega_i$$

Define a rectangular grid on each reference block.
The mappings define a logically rectangular, multiblock grid.



Multiblock gridding strategy

Advantages:

- Flexibility in describing irregular geometries
- Convenience in constructing grids locally
- Local (block) and interface adaptivity
- Modeling faults in aquifers and reservoirs
- Modeling interfaces between rock strata
- Modeling different physics in different blocks
- Suitable for parallel domain decomposition implementation

Difficulties:

- Imposing matching conditions across interfaces in an accurate and stable way
- Developing efficient nonlinear interface solvers

Two-phase immiscible flow model

$$\frac{\partial(\phi s_i \rho_i)}{\partial t} + \nabla \cdot \rho_i \mathbf{u}_i = q_i$$

$$\mathbf{u}_i = -\frac{k_i(s_i)K}{\mu_i}(\nabla p_i - \rho_i g \nabla D)$$

$$s_w + s_n = 1$$

$$p_c(s_w) = p_n - p_w$$

$i = w$ (wetting), n (non-wetting) - phase	s_i - phase saturation
ρ_i - phase density	p_i - phase pressure
ϕ - porosity	q_i - source term
K - absolute permeability tensor	$\mathbf{u}_i$ - Darcy velocity
$k_i(s_i)$ - phase relative permeability	μ_i - phase viscosity
g - gravitational constant	D - depth

Mixed finite element methods

- Simultaneous (accurate) approximation of pressure and velocity
- Local mass conservation: for each element E,

$$\int_{\partial E} \mathbf{u} \cdot \nu = \int_E \nabla \cdot \mathbf{u} = \int_E q$$

- Continuity of normal flux across element faces: for each $e = \partial E_1 \cap \partial E_2$,

$$\int_e (\mathbf{u}|_{E_1} \cdot \nu - \mathbf{u}|_{E_2} \cdot \nu) = 0$$

- Implemented as cell-centered finite differences

Single phase flow model problem

Ω - bounded domain in $\mathbf{R}^d$, $d = 2, 3$
$\bar{\Omega} = \cup_{i=1}^{n} \bar{\Omega}_i$
$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$

$$
\begin{aligned}
\mathbf{u} &= -K\nabla p && \text{in } \Omega \\
\nabla \cdot \mathbf{u} &= q && \text{in } \Omega \\
\mathbf{u} \cdot \nu &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Multiblock domain formulation

On each block Ω_i:

$$\begin{aligned} \mathbf{u} &= -K\nabla p && \text{in } \Omega_i \\ \nabla \cdot \mathbf{u} &= q && \text{in } \Omega_i \\ \mathbf{u} \cdot \nu &= 0 && \text{on } \partial\Omega_i \cap \partial\Omega \end{aligned}$$

On each interface Γ_{ij}:

$$\begin{aligned} p_i &= p_j && \text{on } \Gamma_{ij} \\ [\mathbf{u} \cdot \nu]_{ij} &= 0 && \text{on } \Gamma_{ij} \end{aligned}$$

where

$$\begin{aligned} p_i &= p|_{\partial\Omega_i} \\ [\mathbf{u} \cdot \nu]_{ij} &= \mathbf{u}|_{\Omega_i} \cdot \nu - \mathbf{u}|_{\Omega_j} \cdot \nu \end{aligned}$$

Multiblock discretization

On each block Ω_i:

Mixed finite element (cell-centered finite difference) approximation $(\mathbf{u}_h, p_h)$.

On each interface Γ_{ij}:

Mortar finite element discretization $M_{h,ij}$ on an interface grid.

Interface pressure is approximated by λ_h in the mortar space $M_{h,ij} \Rightarrow$ strong pressure continuity.

Flux continuity is imposed weakly:

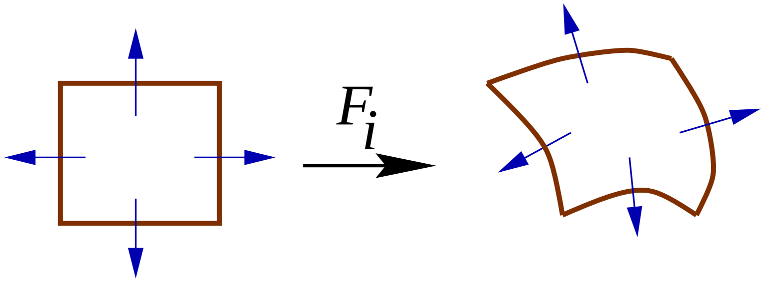
$$\int_{\Gamma_{ij}} [\mathbf{u} \cdot \nu]_{ij} \mu \, d\sigma = 0, \quad \forall \mu \in M_{h,ij}.$$

Lowest order Raviart-Thomas spaces on Ω_i

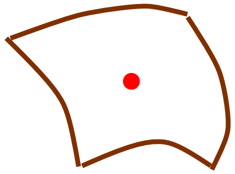
$$\Omega_i = F_i(\hat{\Omega}_i), \quad (DF_i)_{kl} = \partial F_{i,k}/\partial \hat{x}_l, \quad J_i = |\det(DF_i)|$$

Velocity:

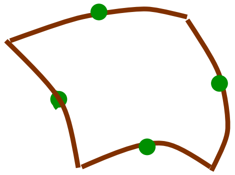
$$\hat{\mathbf{V}}_{h,i} = \left\{ \hat{\mathbf{v}}|_{\hat{E}} = \begin{pmatrix} \alpha_1 + \beta_1 \hat{x}_1 \\ \alpha_2 + \beta_2 \hat{x}_2 \end{pmatrix} : \nabla \cdot \hat{\mathbf{v}} \in L^2(\hat{\Omega}_i), \hat{\mathbf{v}} \cdot \hat{\nu}|_{\partial\hat{\Omega}} = 0 \right\}, \quad \mathbf{V}_{h,i} = \frac{1}{J_i} DF_i \, \hat{\mathbf{V}}_{h,i}$$



Pressure: $W_{h,i} = \{ w|_E = \alpha \}$



Lagrange multiplier pressures: $\Lambda_{h,i} = \{ \mu|_e = \alpha \}$

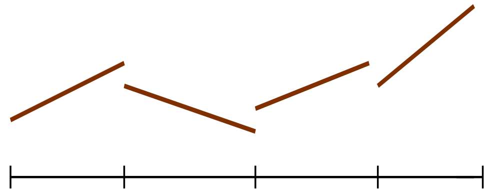


Mortar finite element spaces

Matching interface Γ_{ij}: Use the standard Lagrange multiplier space.

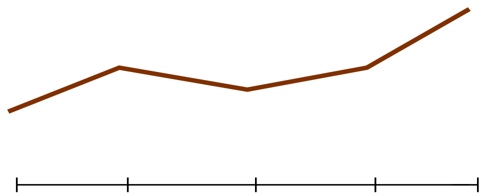
Non-matching interface Γ_{ij}: The piese-wise constant Lagrange multiplier space leads to only sub-optimally convergent method.
On a logically rectangular mesh on Γ_{ij}, possibly different from the meshes on Ω_i and Ω_j, define

$$M^d_{h,ij} = \{\mu \in L^2(\Gamma_{ij}) : \mu|_e \text{ is (bi)linear}\}$$



and

$$M^c_{h,ij} = M^d_{h,ij} \cap C^0(\Gamma_{ij}).$$



$M_{h,ij} = M^d_{h,ij}$ or $M^c_{h,ij}$ - a mortar finite element space on Γ_{ij}.

An expanded mortar mixed finite element method

For $G = J_i (DF_i^{-1})^T DF_i^{-1}$, let

$$G\tilde{\mathbf{u}} = -\nabla p \quad \xrightarrow{F^{-1}} \quad \hat{\tilde{\mathbf{u}}} = \hat{\nabla}\hat{p},$$

$$G^T\mathbf{u} = G^T K G\tilde{\mathbf{u}} \quad \xrightarrow{F^{-1}} \quad \hat{\mathbf{u}} = \mathcal{K}\hat{\tilde{\mathbf{u}}}, \quad \mathcal{K} = J\,DF^{-1} K (DF^{-1})^T.$$

Find $\hat{\mathbf{u}}_h|_{\hat{\Omega}_i} \in \hat{\mathbf{V}}_{h,i}$, $\hat{\tilde{\mathbf{u}}}_h|_{\hat{\Omega}_i} \in \hat{\mathbf{V}}_{h,i}$, $\hat{p}_h|_{\hat{\Omega}_i} \in \hat{W}_{h,i}$, $\hat{\lambda}_h|_{\hat{\Gamma}_{ij}} \in \hat{M}_{h,ij}$ such that

$$\int_{\hat{\Omega}_i} \hat{\nabla}\cdot\hat{\mathbf{u}}_h\, w\, d\hat{x} = \int_{\hat{\Omega}_i} \hat{q} J\, w\, d\hat{x}, \qquad w \in \hat{W}_{h,i},$$

$$\int_{\hat{\Omega}_i} \hat{\tilde{\mathbf{u}}}_h \cdot \hat{\mathbf{v}}\, d\hat{x} = \int_{\hat{\Omega}_i} \hat{p}_h \hat{\nabla}\cdot\hat{\mathbf{v}}\, d\hat{x} - \int_{\partial\hat{\Omega}_i} \hat{\lambda}_h\, \hat{\mathbf{v}}\cdot\hat{\nu}\, d\hat{\sigma}, \qquad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_{h,i},$$

$$\int_{\hat{\Omega}_i} \hat{\mathbf{u}}_h \cdot \hat{\mathbf{v}}\, d\hat{x} = \int_{\hat{\Omega}_i} \mathcal{K}\hat{\tilde{\mathbf{u}}}_h \cdot \hat{\mathbf{v}}\, d\hat{x}, \qquad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_{h,i},$$

$$\int_{\hat{\Gamma}_{ij}} [\hat{\mathbf{u}}_h \cdot \hat{\nu}]_{ij}\, \hat{\mu}\, d\hat{\sigma} = 0, \qquad \hat{\mu} \in \hat{M}_{h,ij}.$$

Cell-centered and face-centered finite differences

Employ the trapezoidal quadrature rule for approximating

$$\int_{\hat{\Omega}_i} \hat{\tilde{\mathbf{u}}}_h \cdot \hat{\mathbf{v}}\, d\hat{x}, \quad \int_{\hat{\Omega}_i} \hat{\mathbf{u}}_h \cdot \hat{\mathbf{v}}\, d\hat{x}, \text{ and } \int_{\hat{\Omega}_i} \mathcal{K} \hat{\tilde{\mathbf{u}}}_h \cdot \hat{\mathbf{v}}\, d\hat{x}.$$

Directly eliminate $\hat{\tilde{\mathbf{u}}}_h$ and $\hat{\mathbf{u}}_h$ on a subdomain.

Obtain finite difference scheme for $\hat{p}_h$ at the cell centers and averages of $\hat{\lambda}_h$ at the centers of the faces on a subdomain boundary.

Stencil: 9 points if $d = 2$ and 19 points if $d = 3$.

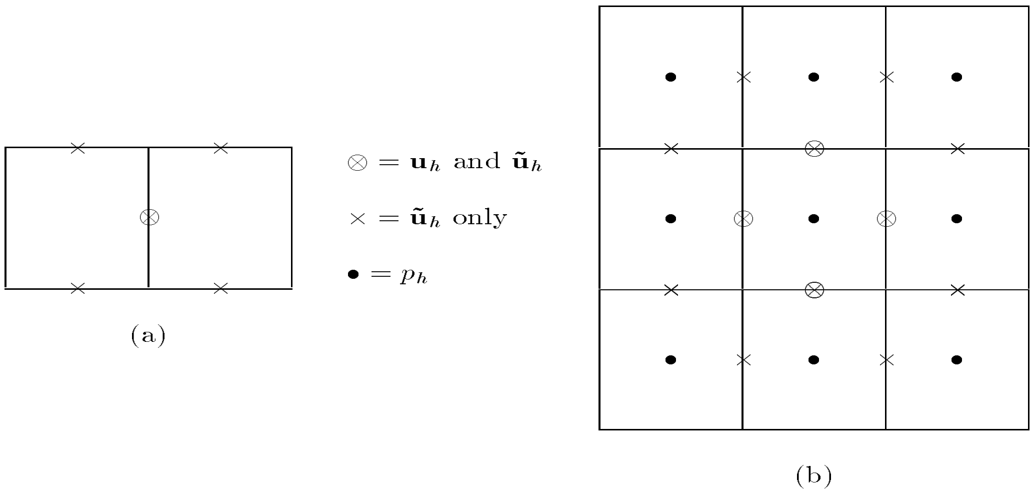


Fig. 1. Nodal points for the standard basis functions.
(a) Stencil for the dependence of $\mathbf{u}_h$ on $\tilde{\mathbf{u}}_h$ ($d = 2$).
(b) Stencil for the pressure p_h ($d = 2$).

Error estimates

Let $\mathcal{Q}_{h,i}$ denote the L^2-projection onto $\mathbf{V}_{h,i} \cdot \nu$.

Theorem:
If for any $\mu \in M_{h,ij}$,

$$||\mu||_{\Gamma_{ij}} \leq \hat{C}(||\mathcal{Q}_{h,i}\mu||_{\Gamma_{ij}} + |\mathcal{Q}_{h,j}\mu||_{\Gamma_{ij}}),$$

then the method has a unique solution and

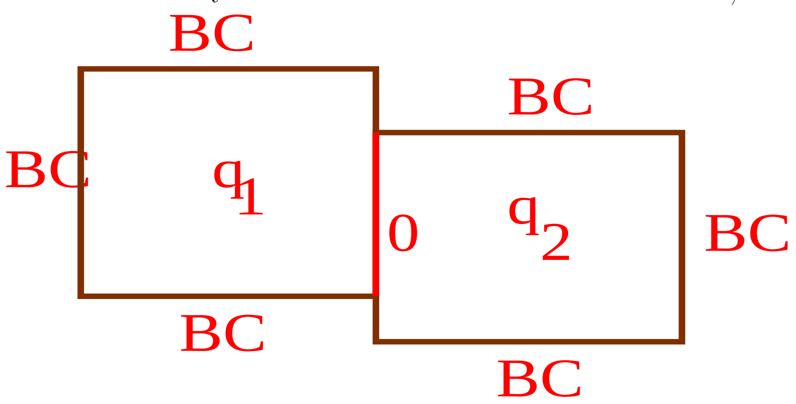
$$\begin{aligned}
&\|p - p_h\| + \|\mathbf{u} - \mathbf{u}_h\| + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\| \leq Ch \\
&|||\mathbf{u} - \mathbf{u}_h||| \leq Ch^{3/2} \\
&|||p - p_h||| + |||\nabla \cdot (\mathbf{u} - \mathbf{u}_h)||| \leq Ch^2,
\end{aligned}$$

where $|| \cdot ||$ denotes the L^2-norm and $||| \cdot |||$ denotes the discrete L^2-norm based on quadrature rule at the nodal points.

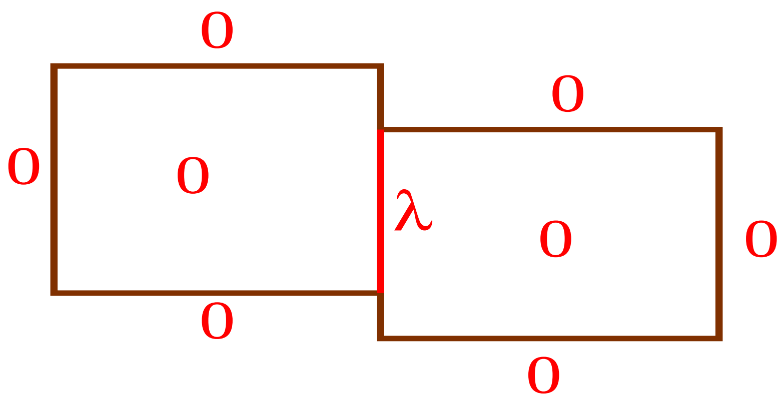
Parallel domain decomposition

Two types of subdomain problems:

$(\bar{\mathbf{u}}_h|_{\Omega_i}, \bar{p}_h|_{\Omega_i})$ is the solution on Ω_i with zero mortar data, correct source and BC's.



$(\mathbf{u}_h^*(\lambda)|_{\Omega_i}, p_h^*(\lambda)|_{\Omega_i})$ is the solution on Ω_i with mortar data $\lambda \in M_h$, zero source and BC's.



Reduction to an interface problem

Define $d_h : M_h \times M_h \to \mathbf{R}$ by

$$d_h(\lambda, \mu) = - \sum_{i,j} \int_{\Gamma_{i,j}} [\mathbf{u}_h^*(\lambda) \cdot \nu]_{ij} \, \mu \, d\sigma$$

Define $g_h : M_h \to \mathbf{R}$ by

$$g_h(\mu) = \sum_{i,j} \int_{\Gamma_{i,j}} [\bar{\mathbf{u}}_h \cdot \nu]_{ij} \, \mu \, d\sigma$$

The solution $(\mathbf{u}_h, p_h, \lambda_h)$ to the original problem satisfies

$$d_h(\lambda_h, \mu) = g_h(\mu), \qquad \mu \in M_h,$$

with

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \quad p_h = p_h^*(\lambda_h) + \bar{p}_h.$$

Iterative interface solver

Lemma
$d_h(\cdot,\cdot)$ is symmetric and positive definite on M_h.

Solve the interface problem

$$d_h(\lambda_h, \mu) = g_h(\mu), \qquad \mu \in M_h$$

in parallel by using preconditioned conjugate gradient (PCG) or multigrid.
Evaluation (on each CG iteration) of the bilinear form

$$d_h(\lambda, \mu) = -\sum_{i,j} \int_{\Gamma_{i,j}} [\mathbf{u}_h^*(\lambda) \cdot \nu]_{ij}\, \mu \, d\sigma :$$

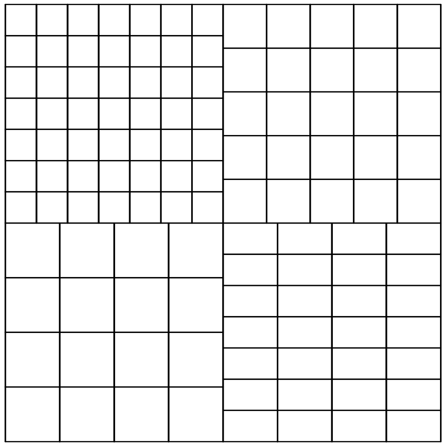
- Project λ onto $\mathbf{V}_{h,i} \cdot \nu$: $\lambda \to \mathcal{Q}_{h,i}\lambda$
- Solve subdomain problems with Dirichlet data $\mathcal{Q}_{h,i}\lambda$ on the interior interfaces to compute the fluxes $\mathbf{u}_h^*(\lambda) \cdot \nu$
- Project the fluxes back to the mortar space M_h to compute the jumps across the interfaces

Computational results

$$p(x,y) = \begin{cases} x^2y^3 + \cos(xy), & 0 \le x \le 1/2, \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos\left(\frac{2x+9}{20}y\right), & 1/2 \le x \le 1, \end{cases}$$

$$K = \begin{cases} I, & 0 \le x < 1/2, \\ 10 * I, & 1/2 < x \le 1, \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = F \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} + \frac{1}{10}\sin(6\hat{x}) \end{pmatrix}.$$

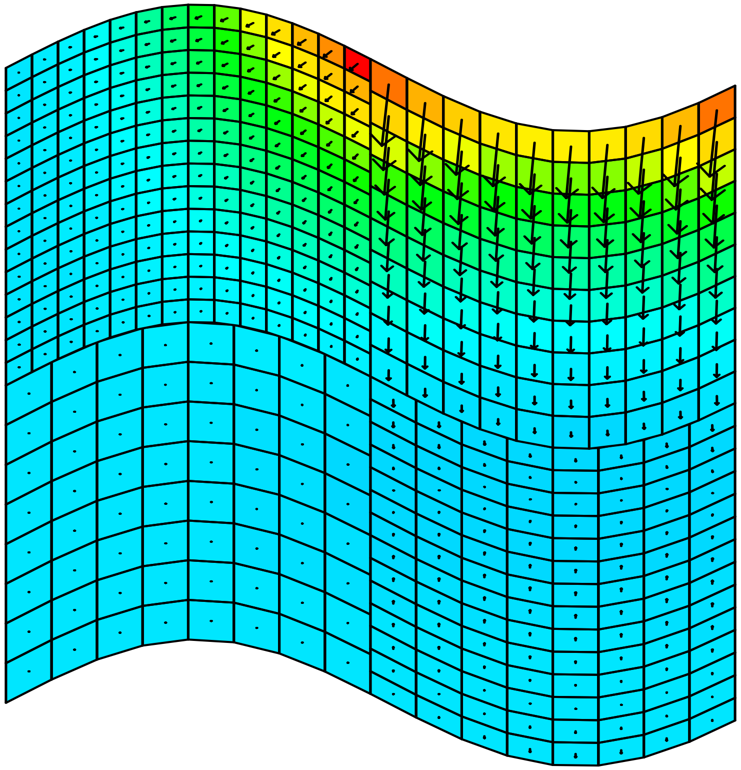


Initial computational grids.

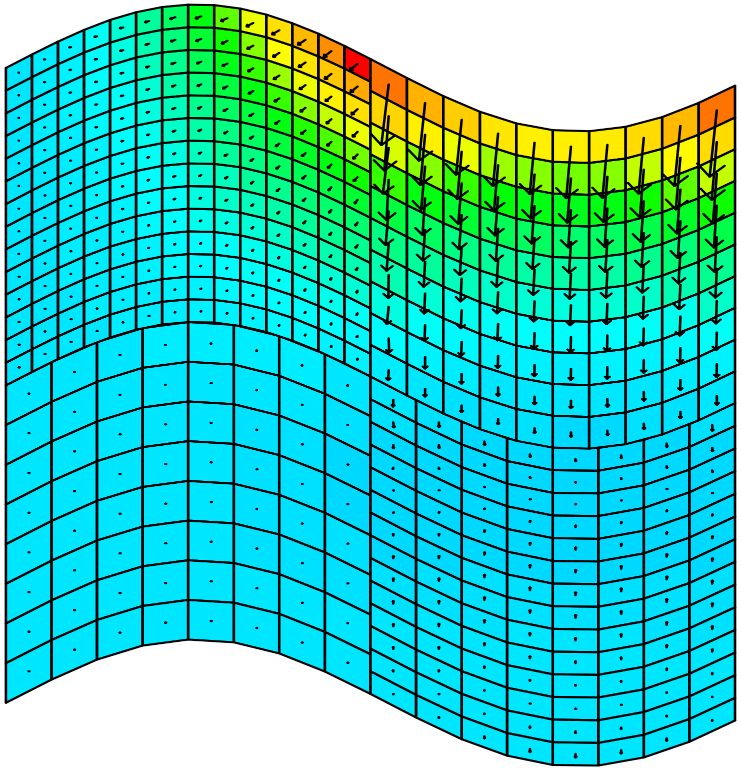
Discrete norm errors and convergence rates

	Continuous mortars			Discontinuous mortars		
$1/h$	$\Vert p-p_h\Vert_{\mathrm{M}}$	$\Vert\mathbf{u}-\mathbf{u}_h\Vert_{\mathrm{M}}$	$\Vert\lambda-\lambda_h\Vert_{\mathrm{M}}$	$\Vert p-p_h\Vert_{\mathrm{M}}$	$\Vert\mathbf{u}-\mathbf{u}_h\Vert_{\mathrm{M}}$	$\Vert\lambda-\lambda_h\Vert_{\mathrm{M}}$
8	6.75E-4	3.05E-2	1.74E-3	7.13E-4	4.60E-2	2.43E-3
16	2.22E-4	1.14E-2	5.43E-4	2.28E-4	1.72E-2	7.01E-4
32	6.42E-5	3.86E-3	1.54E-4	6.50E-5	6.05E-3	1.90E-4
64	1.72E-5	1.29E-3	4.10E-5	1.74E-5	2.10E-3	4.91E-5
1-4	$O(h^{1.77})$	$O(h^{1.53})$	$O(h^{1.80})$	$O(h^{1.79})$	$O(h^{1.49})$	$O(h^{1.88})$
3-4	$O(h^{1.90})$	$O(h^{1.59})$	$O(h^{1.91})$	$O(h^{1.90})$	$O(h^{1.53})$	$O(h^{1.95})$

Computed pressure and velocity

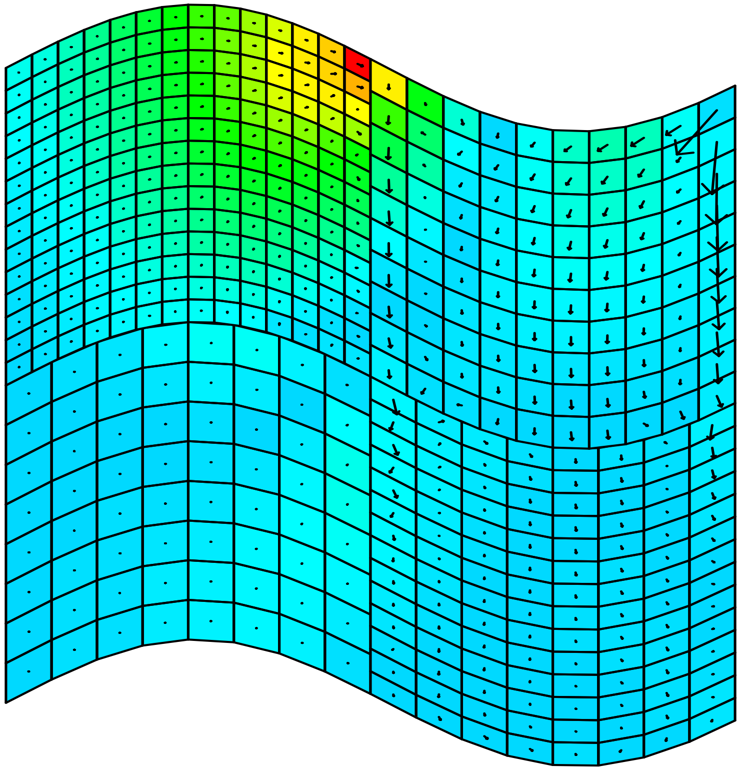


Continuous mortars

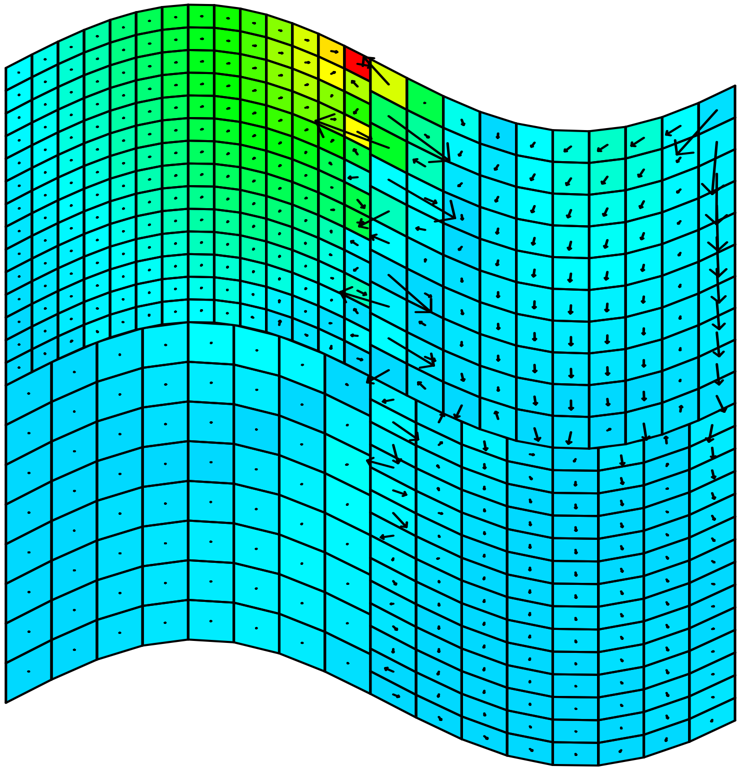


Discontinuous mortars

Pressure and velocity error

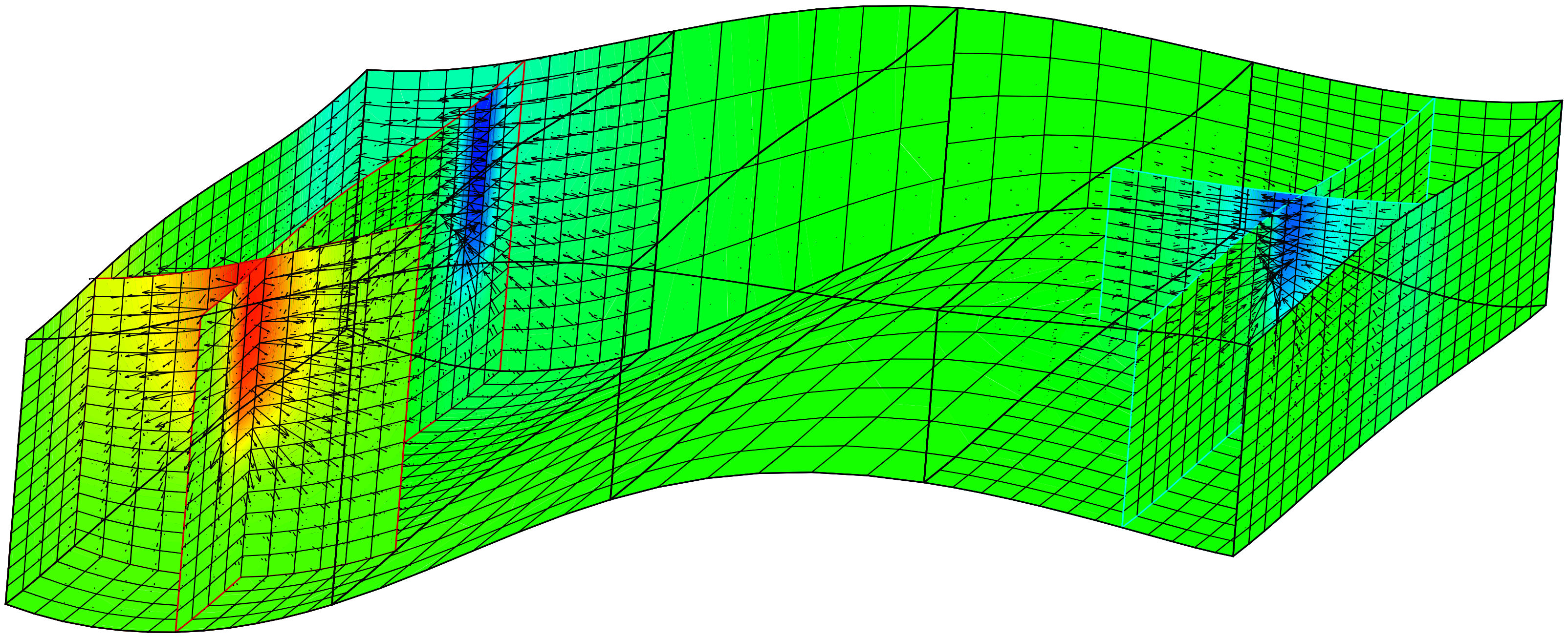


Continuous mortars

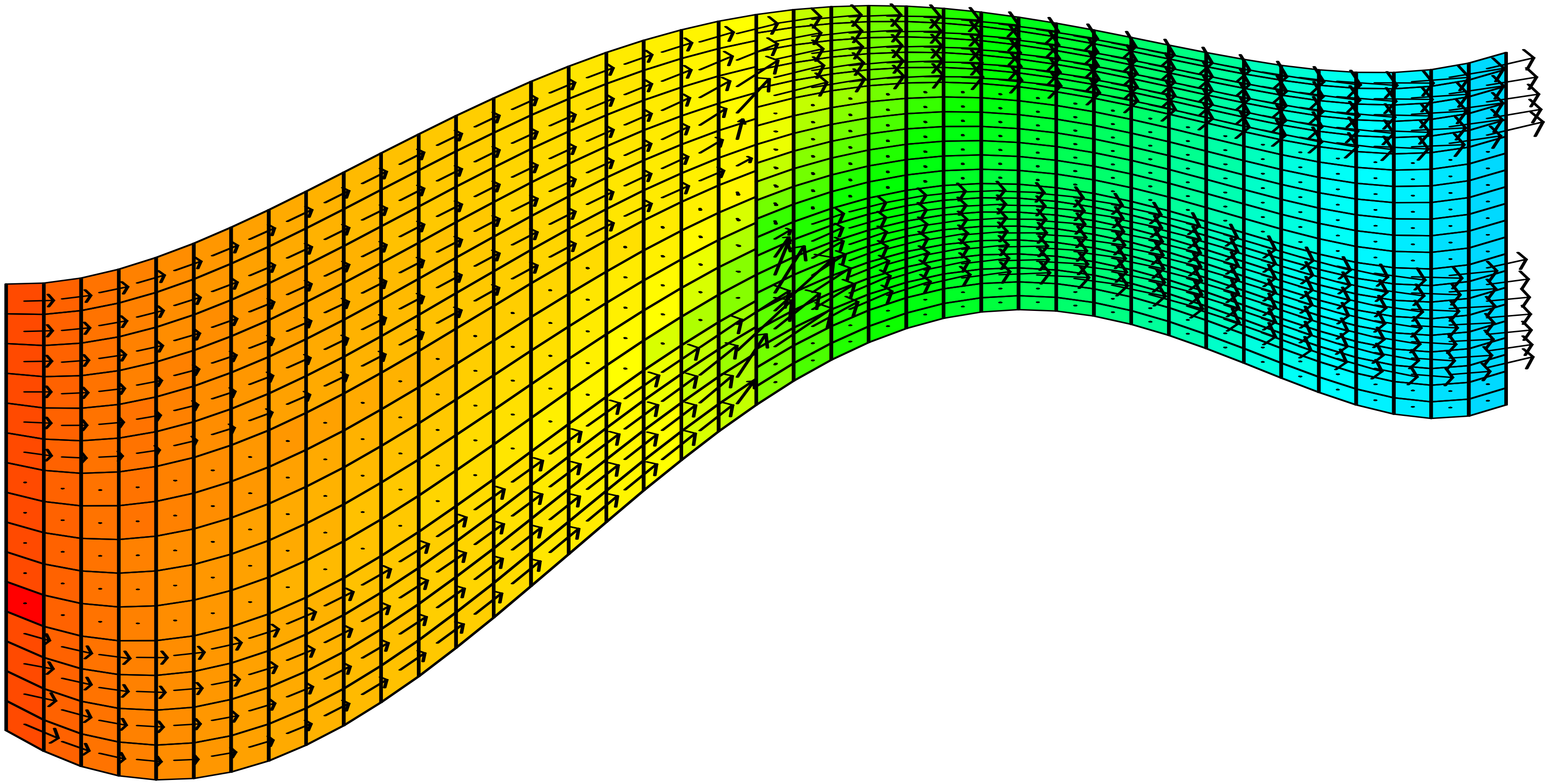


Discontinuous mortars

Flow through reservoir with faults
Pressure and velocity field

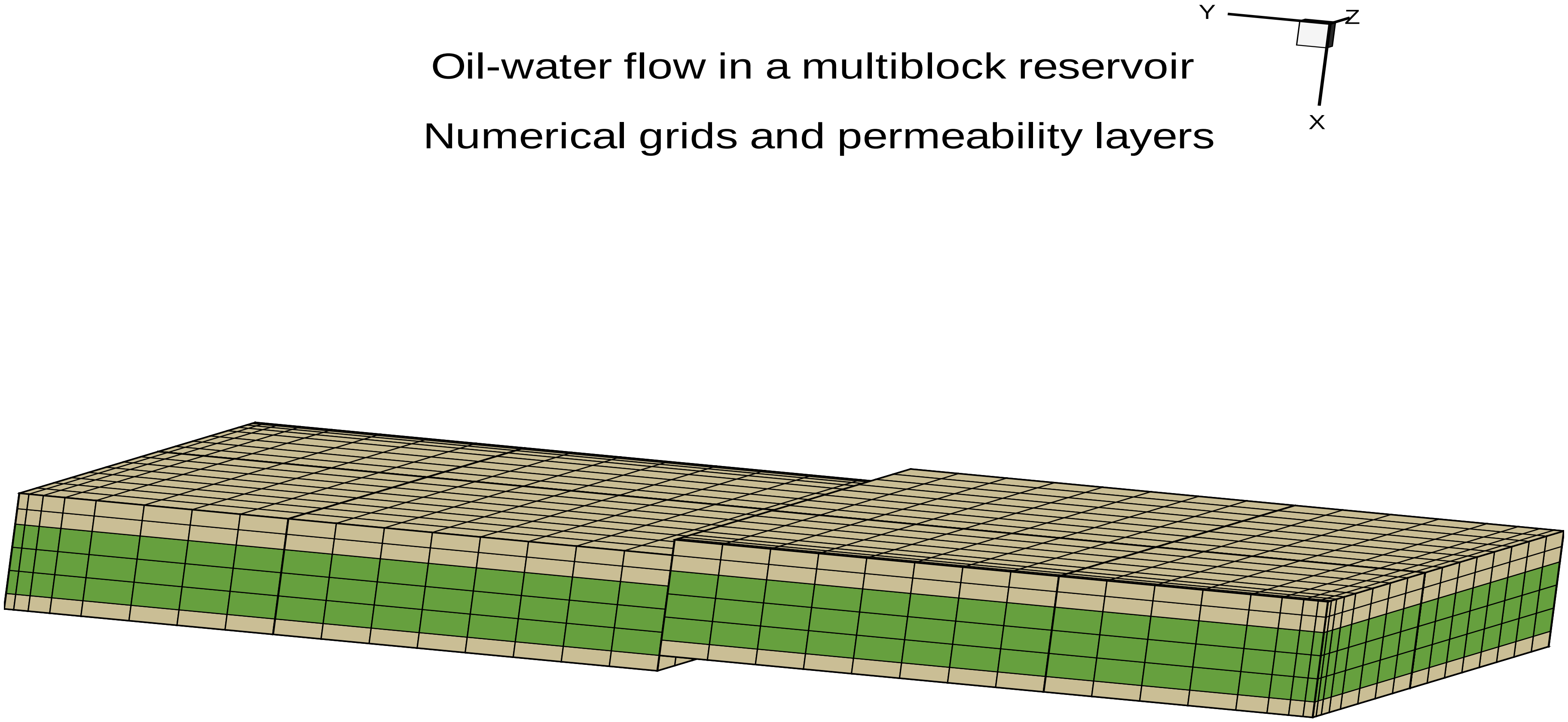


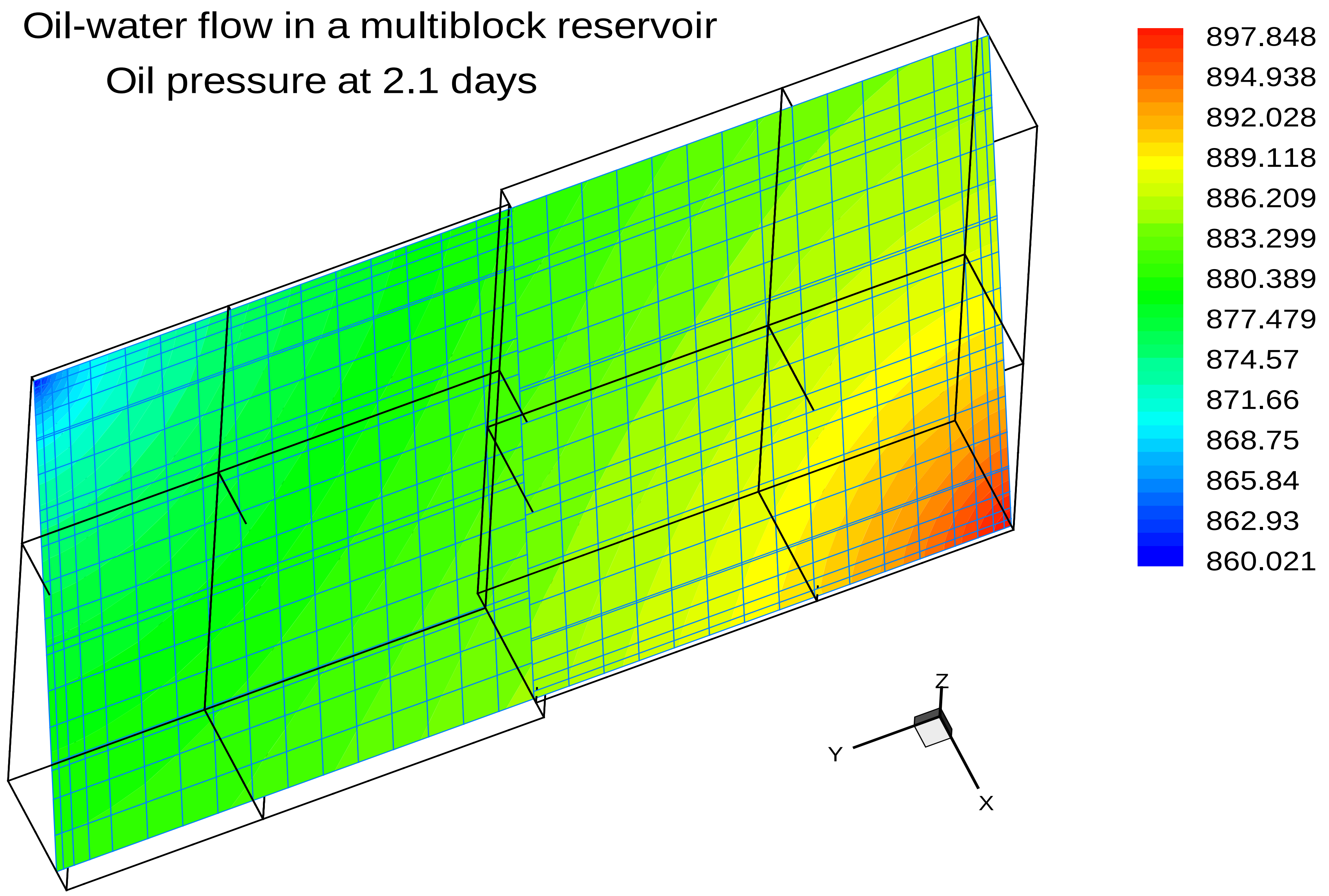
Flow through a layered reservoir with a fault
Pressure and velocity field



Two phase parallel implicit multiblock flow simulator

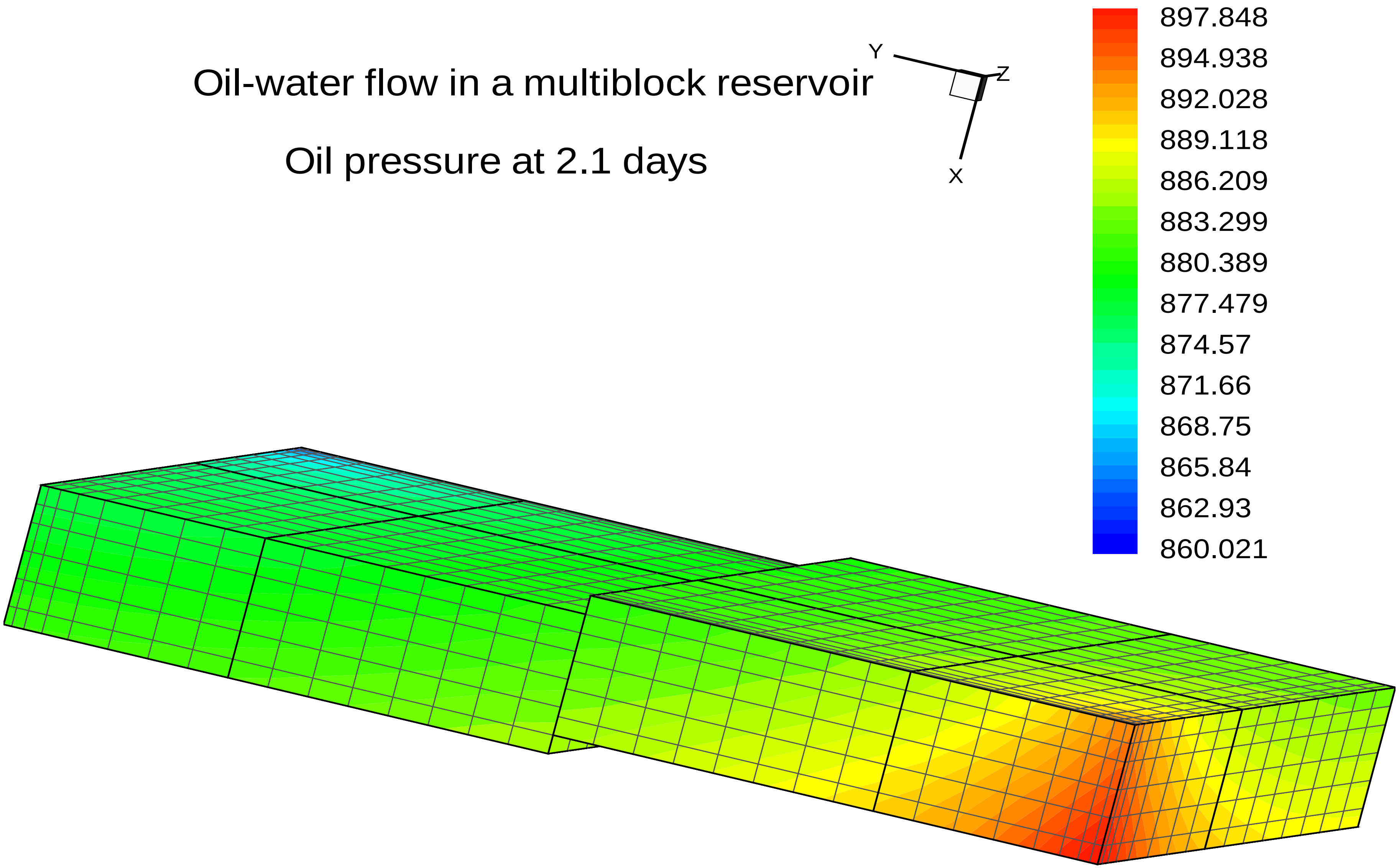
- Irregularly shaped 3D multiblock reservoir
- Logically rectangular grid on each block
- Full tensor permeability
- Slightly compressible fluids
- Primary unknowns - non-wetting phase pressure and concentration
- Conservative cell-centered finite differeneces on each block
- Mortar finite elements for the primary unknowns along non-matching interfaces
- Inexact Newton - forward difference GMRES interface nonlinear iterative solver
- Each block is divided between a number of processors


Oil-water flow in a multiblock reservoir
Numerical grids and permeability layers
Y
Z
X


Oil-water flow in a multiblock reservoir
Oil pressure at 2.1 days
897.848
894.938
892.028
889.118
886.209
883.299
880.389
877.479
874.57
871.66
868.75
865.84
862.93
860.021
Z
Y
X

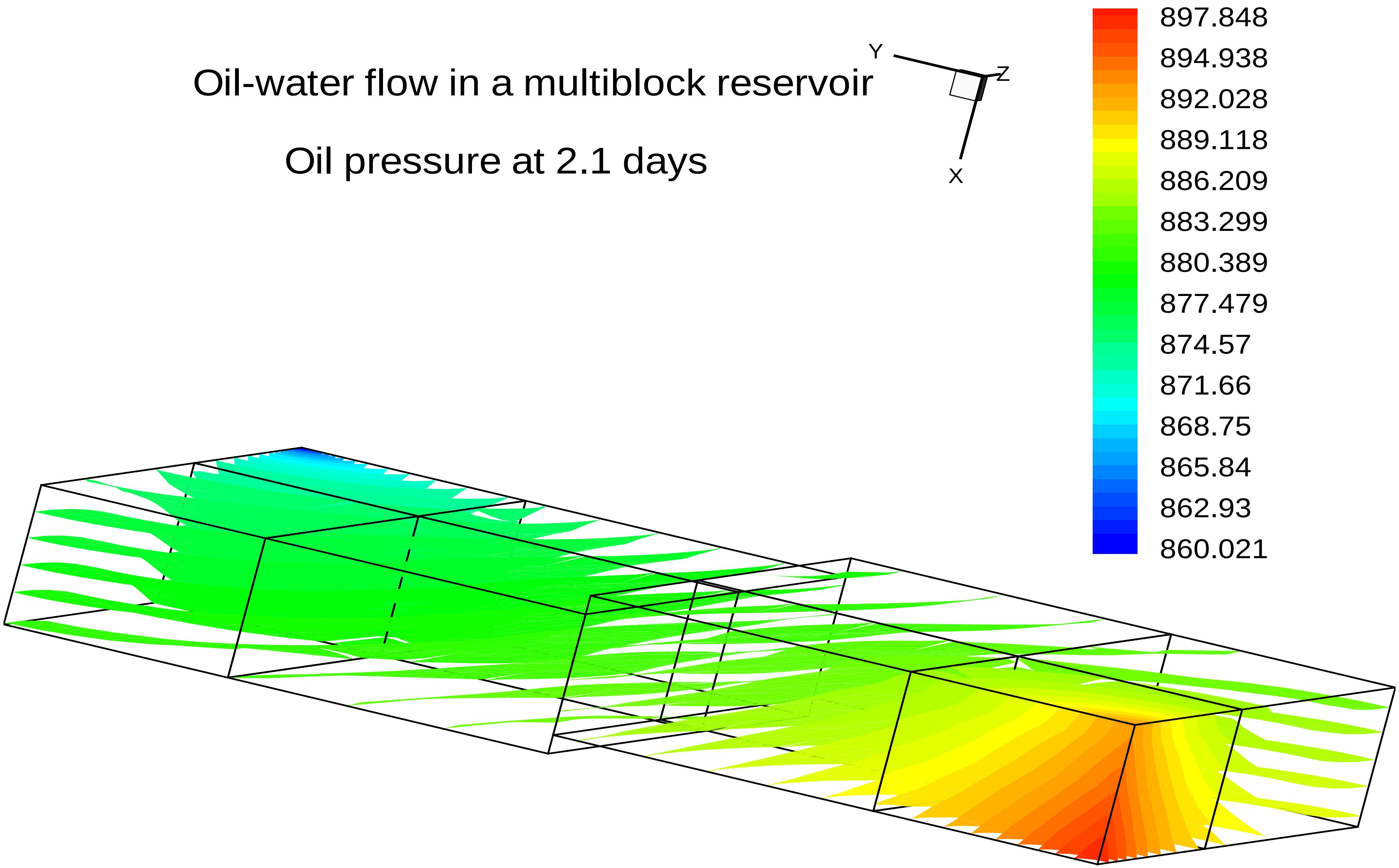
Oil-water flow in a multiblock reservoir

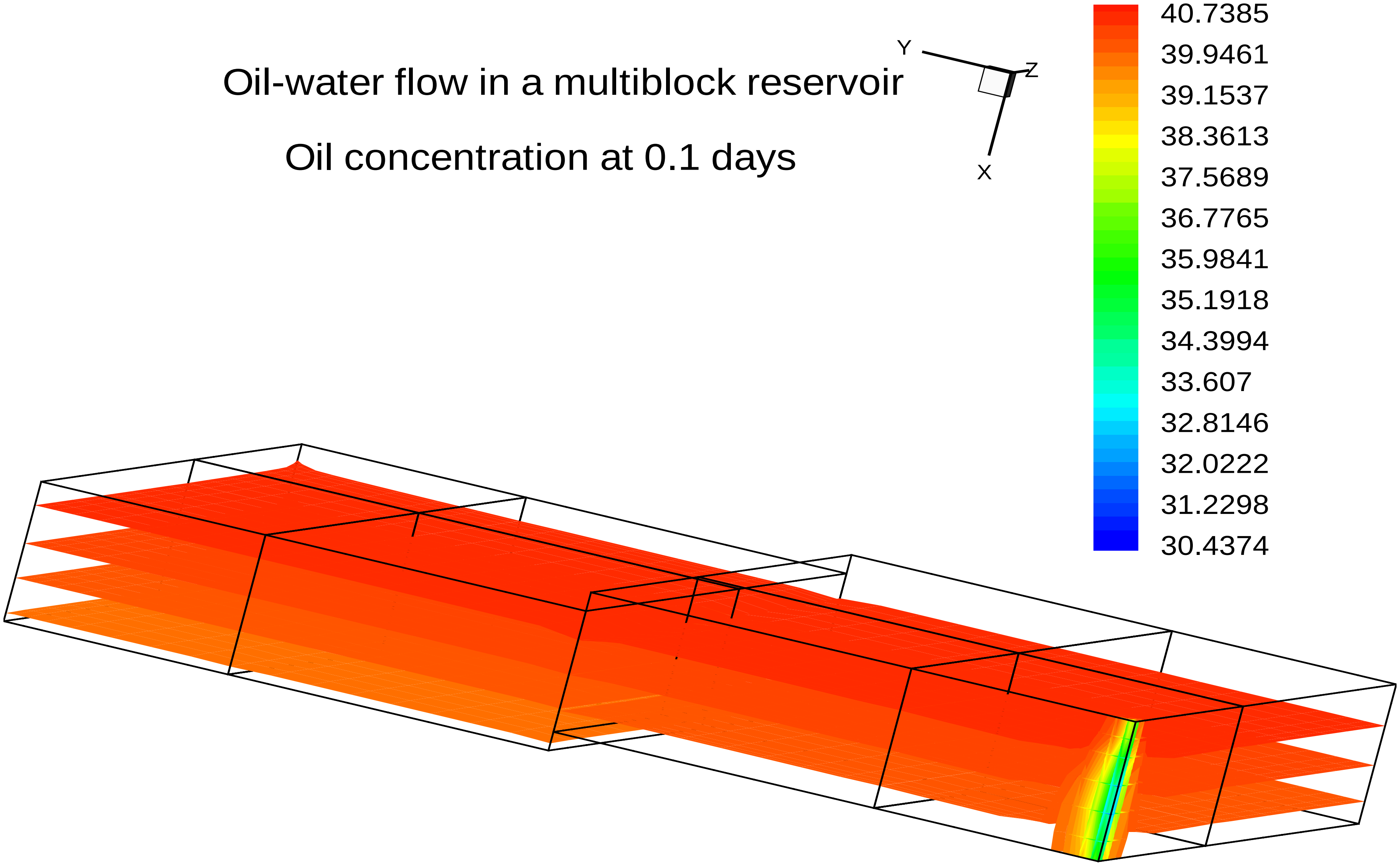
Oil pressure at 2.1 days

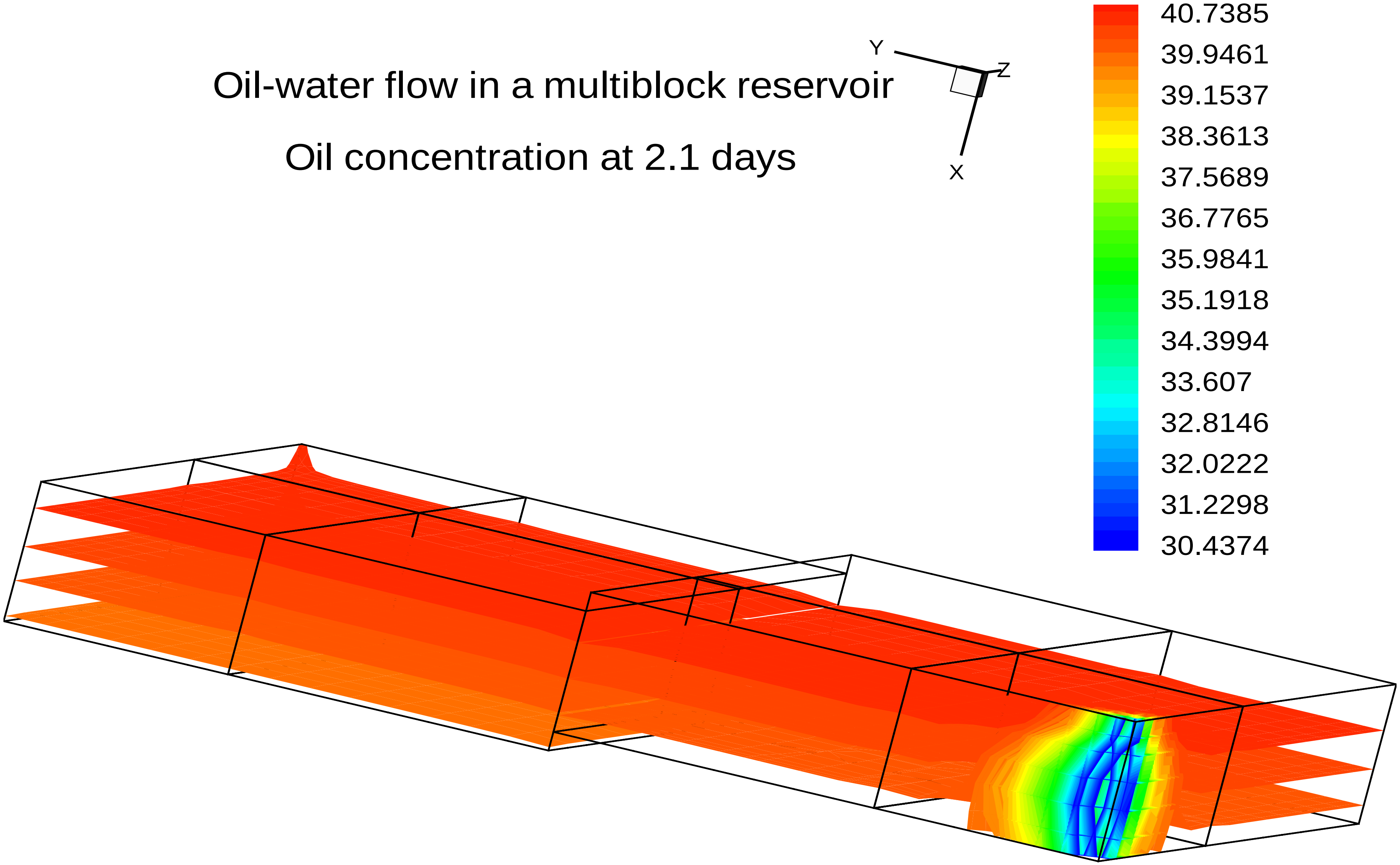


Oil-water flow in a multiblock reservoir

Oil pressure at 2.1 days




Oil-water flow in a multiblock reservoir
Oil concentration at 0.1 days
Y
Z
X
40.7385
39.9461
39.1537
38.3613
37.5689
36.7765
35.9841
35.1918
34.3994
33.607
32.8146
32.0222
31.2298
30.4374


Oil-water flow in a multiblock reservoir
Oil concentration at 2.1 days
Y
Z
X
40.7385
39.9461
39.1537
38.3613
37.5689
36.7765
35.9841
35.1918
34.3994
33.607
32.8146
32.0222
31.2298
30.4374